Equity Implied Vols for All, Part 2: Implied Volatility Curve Design and Fitting

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Outline

- Introduction
- Background, Dividend Modeling
- Vol Curve Design
- Vol Curve Fitting
- Vol Fit Examples
- Conclusion

References and Details:

Pricing Vanilla Options with Cash Dividends (Part 1)

Necessary and Sufficient No-Arbitrage Conditions for the SSVI/S3 Volatility Curve

A General and Practical Volatility Curve Framework (in preparation, 2016)

Mysteries of the (Listed) Equity Options World

- Why do data vendors have different vols for calls and puts, no greeks for many options (ITM, some OTM), and bad implied dividends?
- Why do CNBC, Bloomberg, et al never show sexy vol curves?
- Why does every half-way serious options trading team write their own valuation (pricer, fitter for borrows and vols) and trade analysis infrastructure (mark-ups, PnL decomposition, TCA, etc)?
- Lead-lag relationships in vol space are orders of magnitude slower than in the equity domain.
- Some leading options market makers (OMMs) use hand-calibrated (not auto-fitted) vol surfaces.

Equity Options Market Overview

- In US alone there are 500,000+ options on 4,000+ underliers.
    - Most do not trade on any given day. Most have very wide bid-ask spreads at any given time, especially ITM and very OTM options.
    - Even liquid underliers can have such options.
    - OTOH: Some options have super-tight spreads that are hard to fit with "reasonable" vol skew curves.
- All options can only be valued with real-time, robust implied borrow and sophisticated volatility curves.
- Also required for real-time risk and PnL decomposition.
- Well-designed parametric curves are needed for sensible book-level sensitivities (vanillas + exotics): *normalized vega*, *skew vega*, etc
- All borrow and vol curves are proprietary. Despite big efforts, no data/analytics vendor has them (even EOD historical).

Mysteries.... in more detail

- The equity options market is complicated due to dividends, borrow costs, events, and vol curves with lots of structure – but spreads can be very tight. Nowadays most liquid options (SPY, QQQ, AAPL, VXX) are American-style (much harder than future/index options)
- To value all vanilla options on an equity underlier one needs:
  - Interest rates – freely available (but which one?)
  - Dividend projections – can be bought (but pretty expensive)
  - Borrow curve – not available for purchase at any price
  - Volatility surface, volatility TTX – not available at any price
- To value exotics, one needs an arbitrage-free volatility surface also in the far wings, as input to SLVJ calibrators (not available).
- A dirty secret of the options industry is: Only a handful out of hundreds of players know (sort of):
  - How to properly price with cash divs (model & algo issue)
  - How to imply borrow cost curves
  - How to design and robustly calibrate tradable vol curves in real-time

What do Vanilla Options Market Makers do?

- Use "hacked" Black-Scholes framework for valuation & risk mngt:
  - Pick dividend model, as well as dividend amount, dates.
  - Figure out borrow cost term-structure.
  - Use a volatility TTX (affects relative early-exercise premia).
  - Decide on event weights (underlier specific: FOMC, earnings).
  - Manage or fit implied volatility curves/surface.
  - For greeks use implied vol $\sigma = \sigma(T, K)$ in Black-Scholes model, but correct for spot-vol dynamics (smart delta).
  - Have good and fast underlier valuation.
  - Need fast & robust American pricer (with proper div model).
- Why Black-Scholes?
- Should there be one borrow cost per term?
- Should same implied vol be used for call and put at a given maturity and strike T, K?

Dividend Modeling

- Forty years after Black-Scholes there is no consensus on how to model cash dividends!
- Cash dividends mean that the observed stock price can not follow geometric Brownian motion (GBM).
- A *dividend (pricing) model* is defined by the SDE of the underlier.
- In a vanilla context the question is how to combine the stochastic part of underlier evolution (e.g. *who* follows GBM?) with...
- Three types of dividends:
  - A dividend yield – used to model borrow cost
  - Cash dividends – how most dividends are actually paid
  - Discrete proportional dividends
- Most firms use *blending scheme* to transition from cash dividends on short end to proportional dividends in long term.
- Proportional divs are also useful in times of extreme uncertainty (market-wise or name-specific). E.g. during 2008 crisis.

Dividend Models

- Main two classes of dividend models are:
  - *Spot model*: The dividends come out of the observed stock price. (Need to modify cash dividends at low stock price.)
  - *Hybrid models*: The dividends come out of a "cash buffer", related to the PV of future dividends.
- Spot model might seem naively more reasonable, but in practice leads to a lot of complications and hacks, since not GBM.
- Hybrid models are much simpler to handle for both vanillas and exotics, since *pure stock* (stock ex cash buffer) still follows GBM. Can also easily handle credit risk.
- We will assume a hybrid model from now on.
- NOTE: Even if you care only about European options, dividend modeling matters – how e.g. are SPX and SPY vols related?

Hybrid Models, Notation

- In a hybrid model the stock follows *shifted GBM*, and the prices of (un-discounted) European vanillas for the pure stock are:

$$\hat{C} = +F\,N(d_+) - K\,N(d_-) \quad (1)$$

$$\hat{P} = -F\,N(-d_+) + K\,N(-d_-) \quad (2)$$

- Here $N(x)$ is the normal cdf, log-moneyness $y := \log(K/F)$, and

$$d_\pm := \frac{-y}{\hat{\sigma}} \pm \tfrac{1}{2}\hat{\sigma} \quad , \qquad \hat{\sigma} := \sigma\sqrt{T}$$

- $\sigma = \sigma(T, K)$ is the implied volatility of the option.
- Normalized prices $\hat{V}/F$ are function of two dim-less variables: y, $\hat{\sigma}$.
- Actual prices are obtained by shifting the forward $F = F_T$ and strike K by the shift D_T, that depends on the hybrid model.
- For details see part 1.

Volatility Curve Parametrization Wish List

- Parameters should have simple, intuitive meaning, esp. first three.
- Parameters should be "independent", stable from day to day (parsimonious).
- Little term-structure, if possible.
- No-arbitrage constraints should be "easy" to incorporate.
- Parametric vols should be easy/fast to compute.
- No hacks! (in wings, etc)
- Vol curves arising from standard "SLVJ"-type model should be fittable within a few bps (at worst).

Benefits of good volatility curve/surface parametrization

- Can (pretty) easily be set/checked by humans if necessary/desired.
- With suitable fitting framework, have hope of producing fast and robust implied vol curves/surfaces.
- Portfolio level greeks via parameter bumps make sense for vanillas and exotics, and are east and fast to calculate.
- Local vols can be produced fast & robustly via Dupire formula.
- If we can fit all SLVJ models, can fit all real-world surfaces(?).

Our parametrization approach

- Work one term at a time, impose smoothness across terms.
- Factor out overall vol level (ATF) as: $\sigma_0 := \sigma(T, K = F)$.
- Define "shape" curve $f(z) = f(z|\mathbf{p})$ as function of normalized strike (NS)[1]

$$z := \frac{y}{\hat{\sigma}_0} = \frac{\log(K/F)}{\sigma_0 \sqrt{T}}$$

such that

$$\sigma(z)^2 = \sigma_0^2 \, f(z|\mathbf{p})$$

- There are no standard definitions – we define dimensionless "skew" and "smile/convexity" as slope and curvature of shape curve:

$$f(z) =: 1 + s_2 \, z + \tfrac{1}{2} c_2 \, z^2 + \ldots$$

[1] For hybrid models with $D \neq 0$, use: $K/F \to (K-D)/(F-D)$.

Our parametrization approach (cont'd)

- s_2 and c_2 tend to have mild term-structure; they are even comparable across names. Have been range-bound for decades.
- Sometimes it is useful to work with s_1, c_1 defined via

$$\sigma(z) \;=:\; \sigma_0 \left(1 + s_1 z + \tfrac{1}{2} c_1 z^2 + \ldots\right)$$

- Trivially: $s_2 = 2s_1$, $c_2 = 2(c_1 + s_1^2)$.
- Note that

$$\sigma(z) \;=\; \sigma_0 + \frac{s_1}{\sqrt{T}} \log(K/F) + \ldots,$$

so that an alternative definition of skew

$$\tilde{s}_1 \;:=\; K \frac{\partial \sigma}{\partial K}\Big|_{K=F} \;=\; \frac{s_1}{\sqrt{T}}$$

- No simple relationships between alternative definitions of curvature/convexity/smile.

No-Arbitrage Constraints in Vol-Space

- No butterfly arbitrage: Implied density ρ should be positive

$$\hat{C}(T,K) \;=\; \int_0^\infty dS_T\;(S_T - K)_+\;\rho_T(S_0 \to S_T)$$

$$\Rightarrow \quad \partial_K^2 \hat{C}(T,K) \;=\; \rho_T(S_0 \to S)|_{S=K}$$

- No calendar arbitrage: Total BS variance $w(y) := T\sigma(y)^2$ has to be increasing in T at any fixed y.
- Necessary (but generally not sufficient) constraint on the asymptotic wing behavior of implied vols (R. Lee, 2004):

$$w(y) \le 2|y| \qquad\qquad \text{as} \quad |y| \to \infty$$

Simple consequences: Implied density 1

- Local vols and implied densities can be calculated most neatly in terms of the total variance $w(y) = T\sigma(z)^2$. Eg the implied density:

$$\rho(y) \;=\; \frac{g(y)}{\hat{\sigma}(y)}\; n(d_-(y))\;,$$

where $n(x) = N'(x)$ is the normal density, and

$$g(y) \;=\; \left(1 - \frac{y\,w'(y)}{2w(y)}\right)^2 \;-\; \frac{1}{4}\left(\frac{1}{w(y)} + \frac{1}{4}\right) w'(y)^2 \;+\; \frac{1}{2}\,w''(y)$$

- Absence of butterfly arbitrage: $g(y) \geq 0$ for all y.
- In Black-Scholes case: $g(y) = 1$ for all y.
- NOTE: Will use same symbol whether we consider a quantity a function of z or y.

Simple consequences: Implied density 2

- For our vol curve parametrizations $w(y) = \hat{\sigma}_0^2 \, f(z)$.
- Then $w'(y) = \hat{\sigma}_0 f'(z)$ and $w''(y) = f''(z)$, so that

$$g(y) = \left(1 - \frac{z\, f'(z)}{2 f(z)}\right)^2 - \frac{1}{4}\,\frac{f'(z)^2}{f(z)} - \frac{\hat{\sigma}_0^2}{16}\, f'(z)^2 + \frac{1}{2}\, f''(z) \;=:\; g(z)\,.$$

- The vol level appears in only one place! All else only depends on shape parameters.
- Makes the analysis of butterfly arbitrage significantly simpler (but is still very hard in general).
- Will see example later for S3/SSVI curve.

ATF No-Arbitrage Constraints

- If $w(z) \;=\; \hat{\sigma}_0^2\,(1+s_2 z+\frac{1}{2}c_2 z^2+\ldots)$, then

$$g(z\!=\!0) \;=\; 1 \,+\, \tfrac{1}{2}c_2 \,-\, \tfrac{1}{4}s_2^2\,(1+\tfrac{1}{4}\hat{\sigma}_0^2)$$

- $g(0)\geq 0$ implies upper bound on slope

$$s_2^2 \;\leq\; \frac{4+2c_2}{1+\frac{1}{4}\hat{\sigma}_0^2}$$

or lower bound on curvature ($c_1 = \frac{1}{2}c_2 - \frac{1}{4}s_2^2$)

$$c_1 \;\geq\; -1 \,+\, \tfrac{1}{16}\,s_2^2\,\hat{\sigma}_0^2 \;\approx\; -1$$

- Very relevant around FOMC and earnings where not just $c_1 < 0$ but even $c_2 < 0$ can happen!

Specific Curves: Parabolas

- What are simplest possible curves? Need at least 3 parameters for ATF behavior.
- Vendors often use

$$\sigma(y)^n \ = \ \sigma_0^2 + s\ y + \tfrac{1}{2}c\ y^2 \qquad \text{(or in terms of } z\text{)}$$

- Obviously has arbitrage in wings for $n = 1, 2$.
- Slight hope for $n = 4$, but would imply symmetric wings, which is intuitively and empirically wrong.
- Positivity has to be enforced too.
- Must do better...

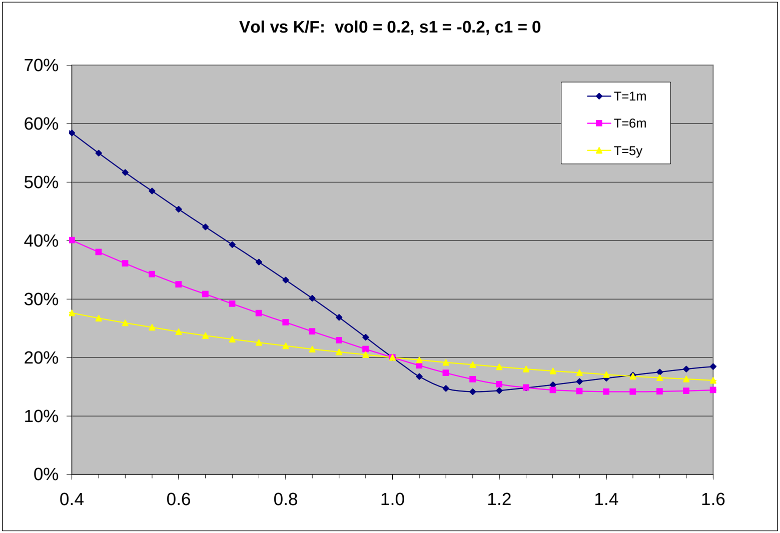
Specific Curves: S3/SSVI

- Simplest sensible curve with 3 parameters ($c_2 \geq 0$):

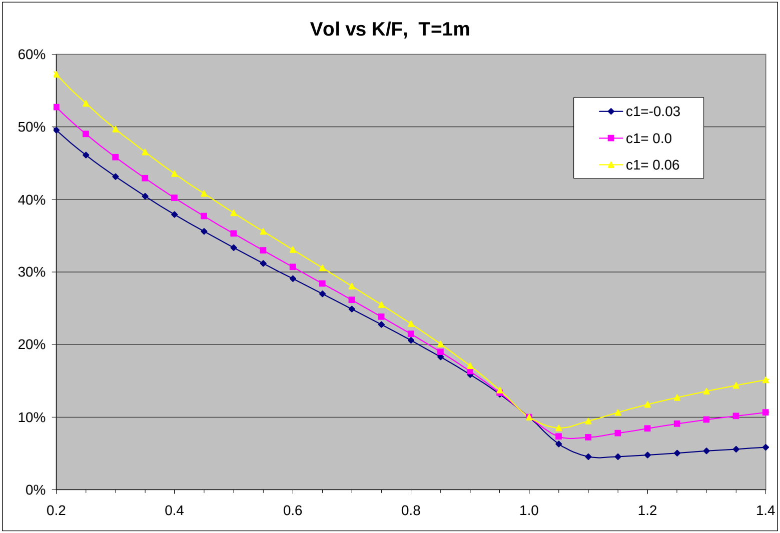
$$\sigma^2(z) \;=\; \sigma_0^2 \left(\tfrac{1}{2}(1 + s_2 z) + \sqrt{\tfrac{1}{4}(1 + s_2 z)^2 + \tfrac{1}{2} c_2 z^2} \right)$$

- Was independently discovered by TRK (2003, "S3") and Gatheral/Jacquier (2013, "SSVI" = Simple SVI).
- Allows surprisingly varied skew shapes, including "takeover-for-cash"-type curves as $c_2 \to 0$. See plots.
- Allows fitting of vast majority of US equity names.
- Relatively easy to avoid (butterfly) arbitrage.
- In fact, in terms of the dimensionless variables $\hat{\sigma}_0, s_2, c_2$ can completely answer the butterfly-arbitrage question...

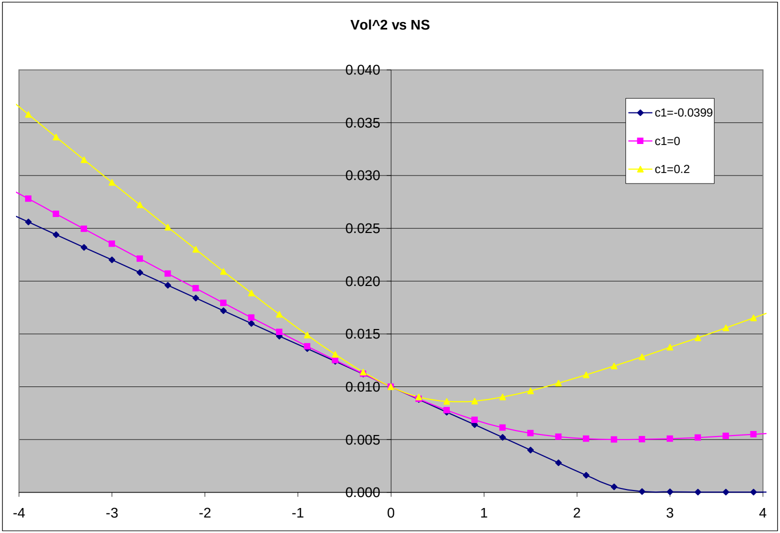
S3 shapes: different terms



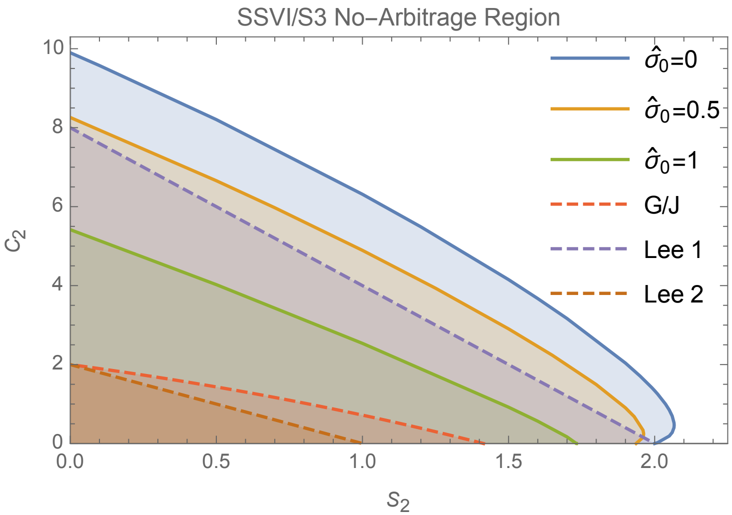
S3 shapes: different curvatures



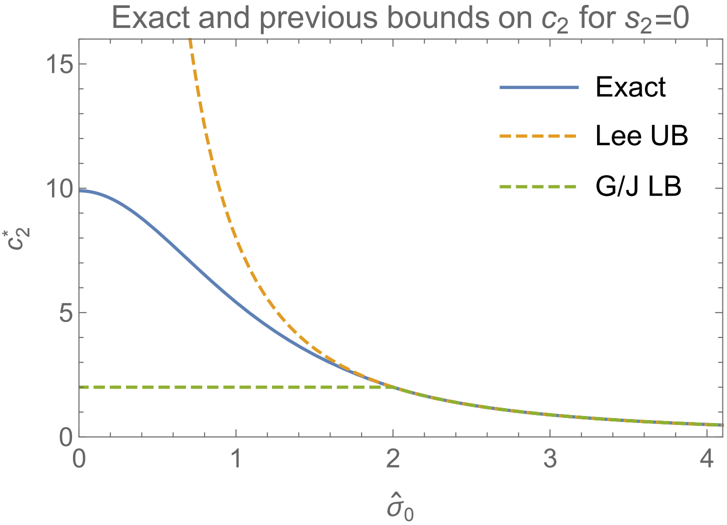
S3 shapes: looked at the right way: σ^2 versus z



Necessary and Sufficient No-Arb Conditions for S3/SSVI



Necessary and Sufficient No-Arb Conditions for S3/SSVI



Specific Curves: 5 Parameters

- Besides 3 parameters for ATF would be nice to have independent parameters $C_\pm$ for wings:

$$\sigma(z)^2 \;\rightarrow\; \sigma_0^2 \; C_\pm \; |z| \qquad \text{as} \quad z \rightarrow \pm\infty \qquad (\hat{\sigma}_0 C_\pm \leq 2)$$

- For S3/SSVI: $\quad C_\pm \;=\; \sqrt{\tfrac{1}{4}s_2^2 + \tfrac{1}{2}c_2} \;\pm\; \tfrac{1}{2}s_2$
- For Jim Gatheral's SVI and others (JW/L5, TRK) the $C_\pm$ are independent parameters.
- Just some algebra to re-express their "raw" parametrization in terms of natural parameters $\sigma_0, s_2, c_2, C_-, C_+$. (Or minimum variance ratio instead of c_2.)
- Can fit some names better than with S3/SSVI.... but surprisingly not much better in many cases!?
- Certainly can not fit W-shaped curves around events (still $c_2 \geq 0$).

Specific Curves: The most liquid names

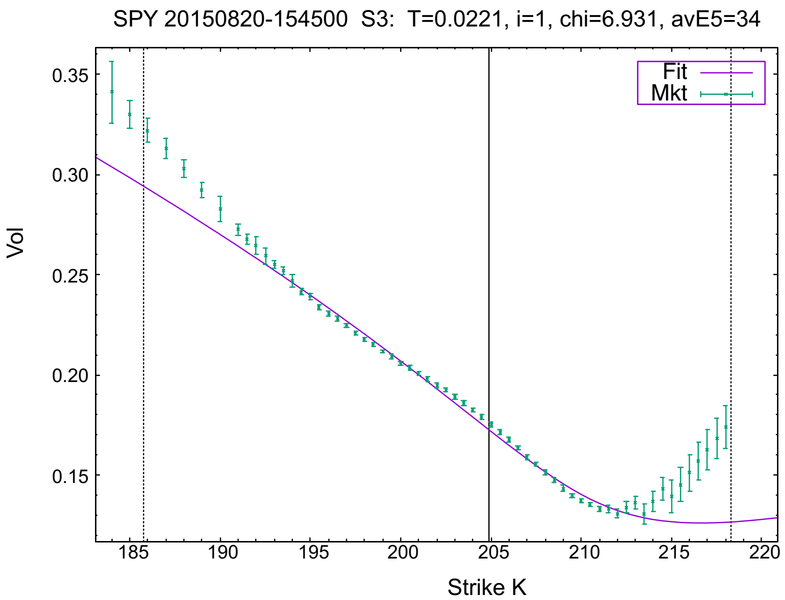
- For very liquid names (SPY, other ETFs, AAPL etc, KOSPI) none of the analytic curves (SVI, L5 or amendments) work well, even in the absence of events.
- There is a fundamental problem with the shapes allowed by these curves: Curvature has unique maximum around ATF, but that's not what the market wants!
- Need more flexible shapes that can handle more generic curvature structures, incl. negative curvature around ATF: C5, C6, C7, C8.

Volatility fitting framework

- Input to fitter are implied vols with error bars (after proper div modeling, borrow implication, etc).
- All our vol curves have sensible dimensionless parameters (first three are universal), which allows the use of curve-independent heuristics from 16 years of vol fitting experience across many names, geographies and asset classes.
- Fit one term at a time, transfer information between terms, for smoothness and stability.
- Minimize chi-square + soft penalties, for robustness and to allow the fitting of terms with less (effective) data than parameters.
- Good microprices help, but even then various heuristics are needed to deal with data issues in real-time.
- Keeping track of quality-of-fit metrics and error bars for final outputs is crucial for real-time trading applications.

Volatility fitting examples

- Examples are fits of American-style options on liquid US ETFs or stocks (plus E-mini futures options).
- Starting with options and underlier prices, we need to
    - pick interest rate
    - pick cash divs (if appropriate)
    - imply borrow cost for each term to get "American PCP"
    - imply vol-by-strike,
    - fit all terms to various vol curves
- Are using simple mid for prices; vol error bars come from bid-ask spread in price space.
- Equity option price data were provided by MayStreet LLC.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Fit
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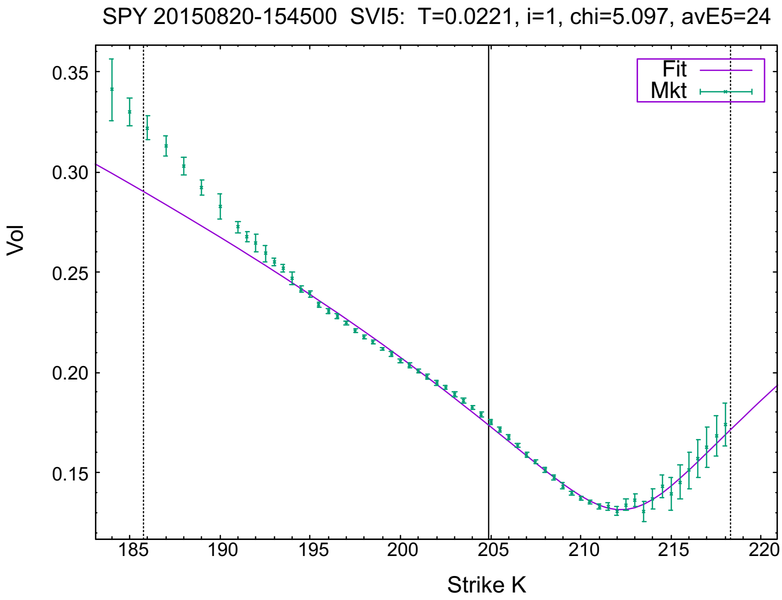
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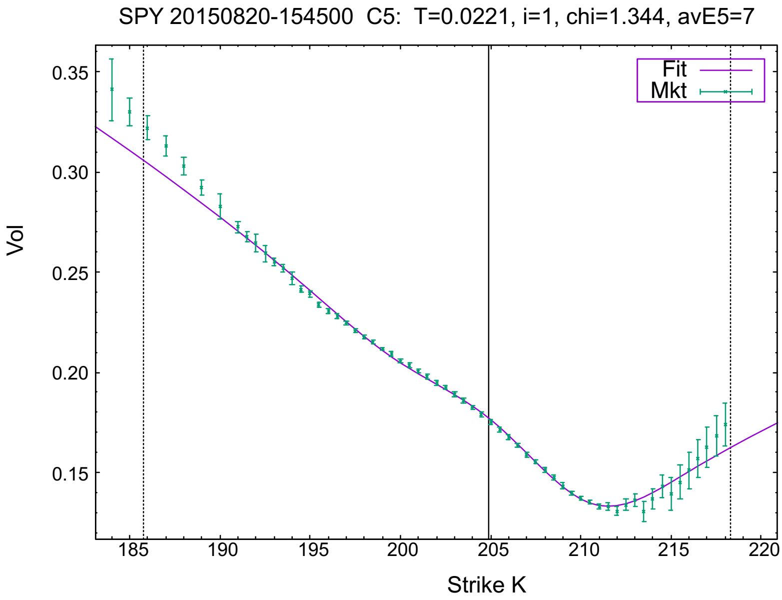
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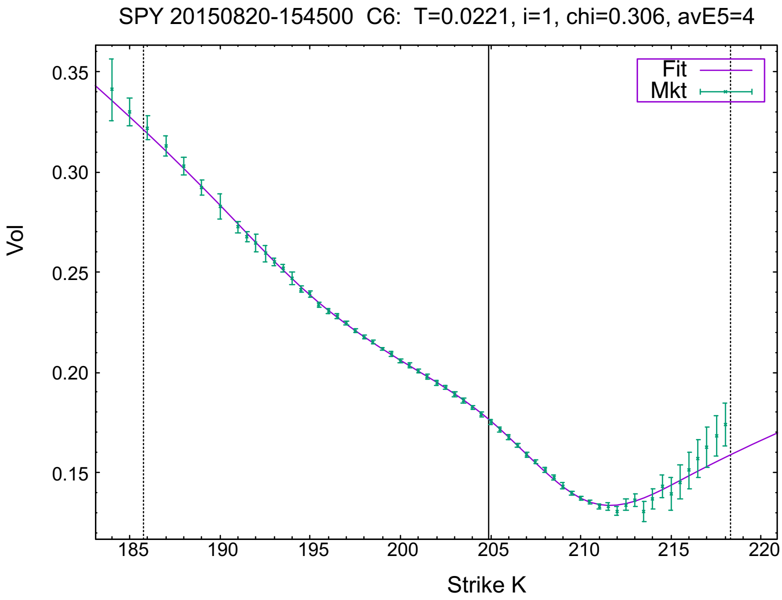
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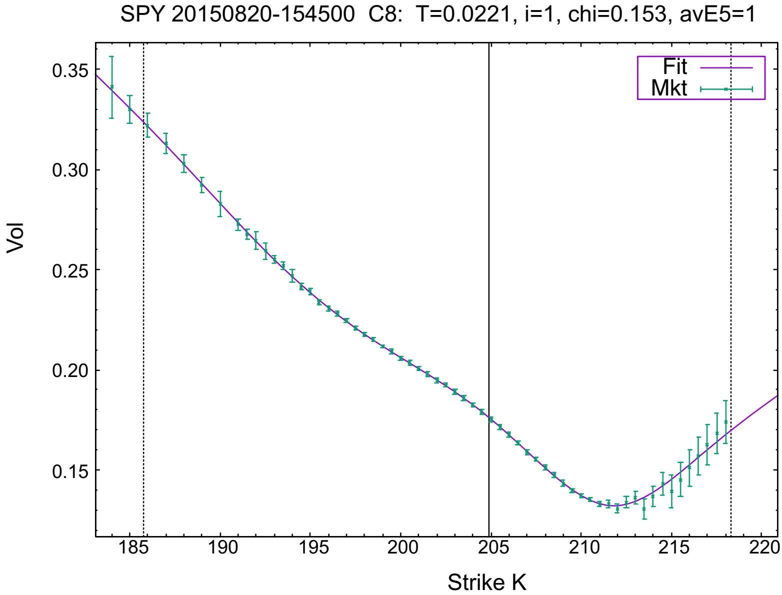
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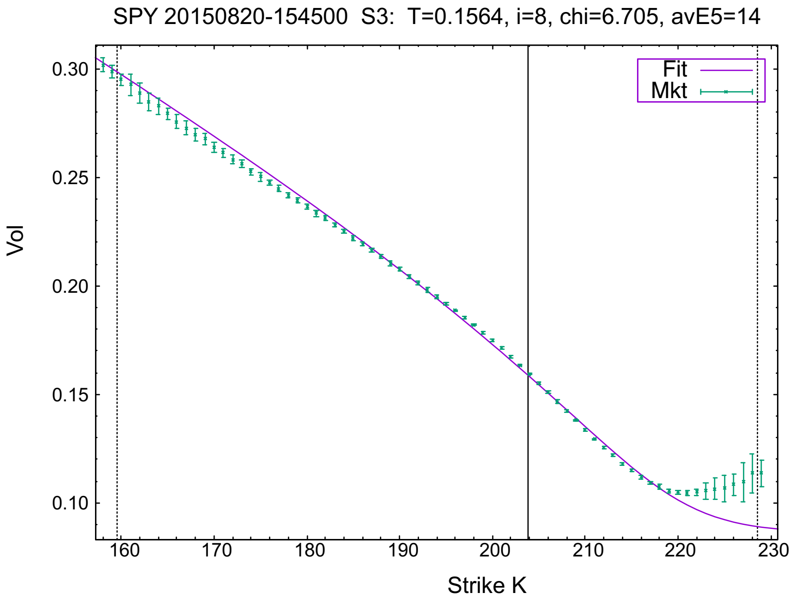
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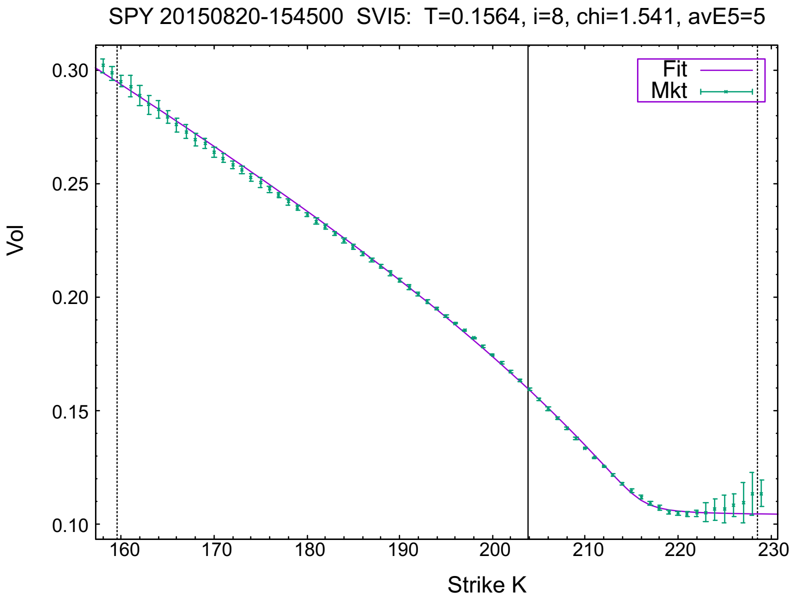
Strike K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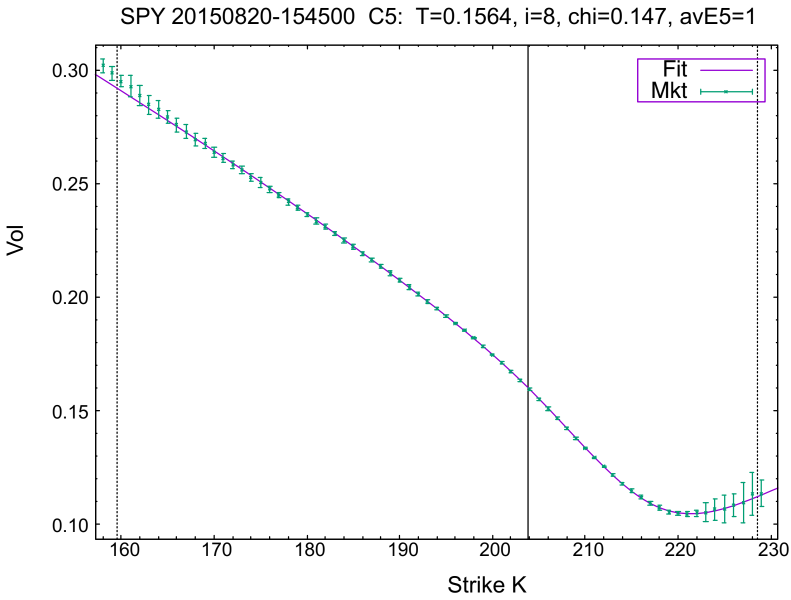
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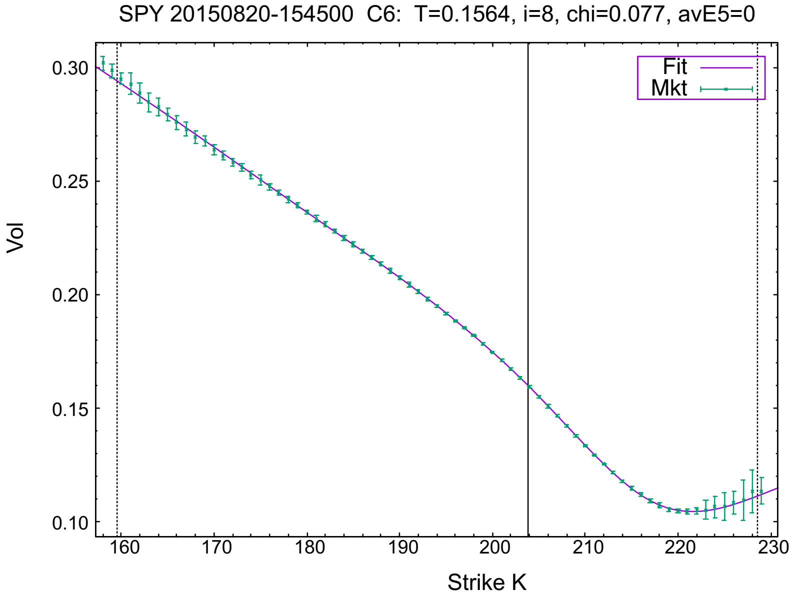
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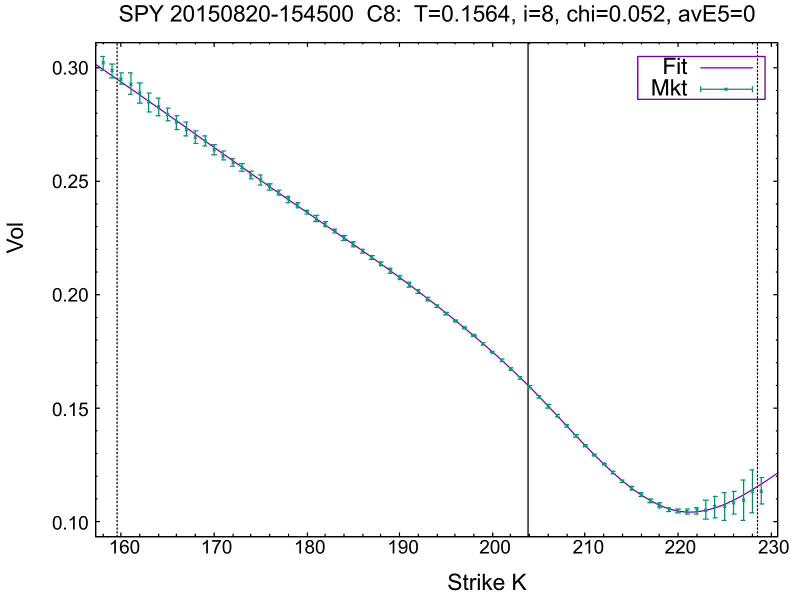
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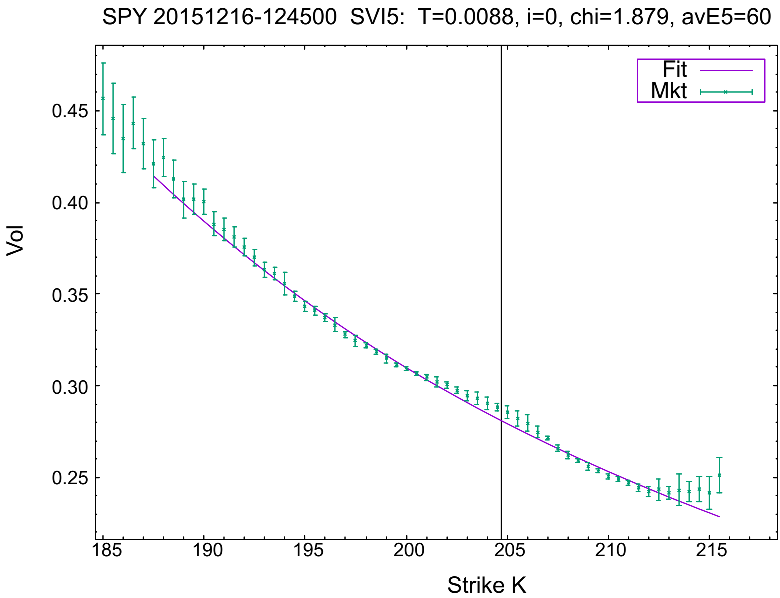
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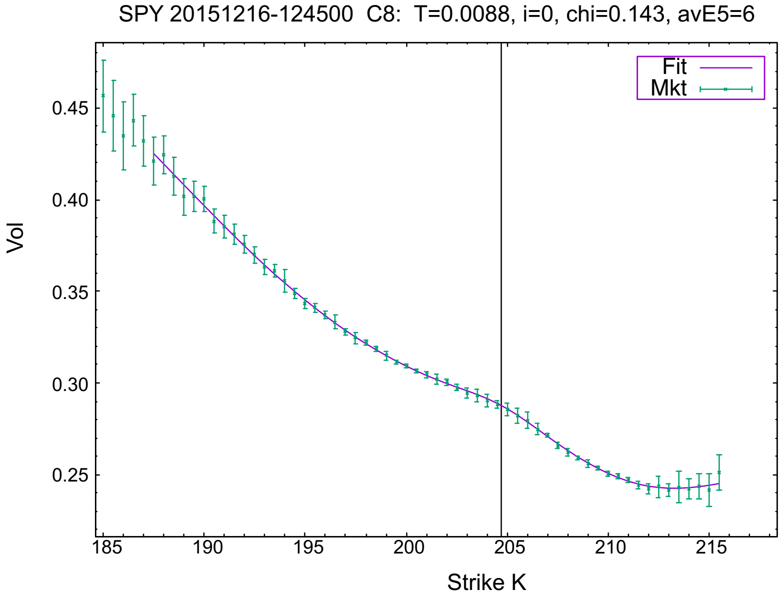
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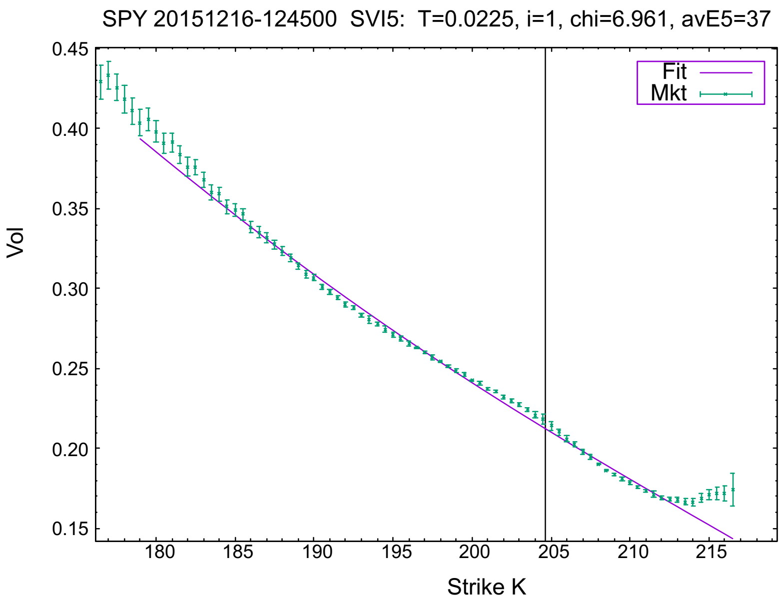
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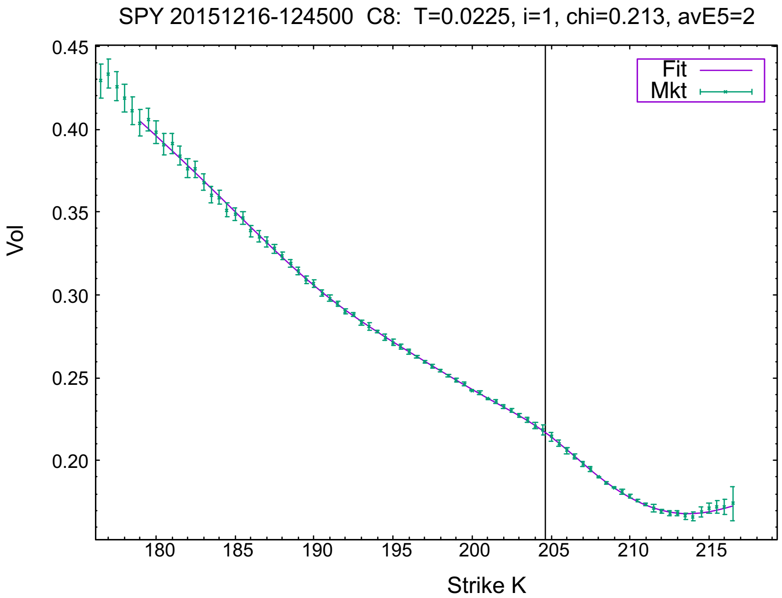
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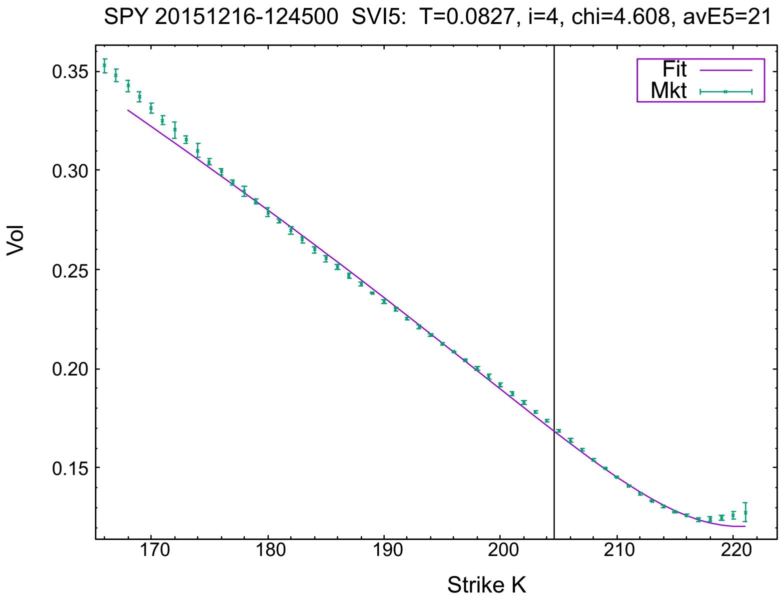
Strike K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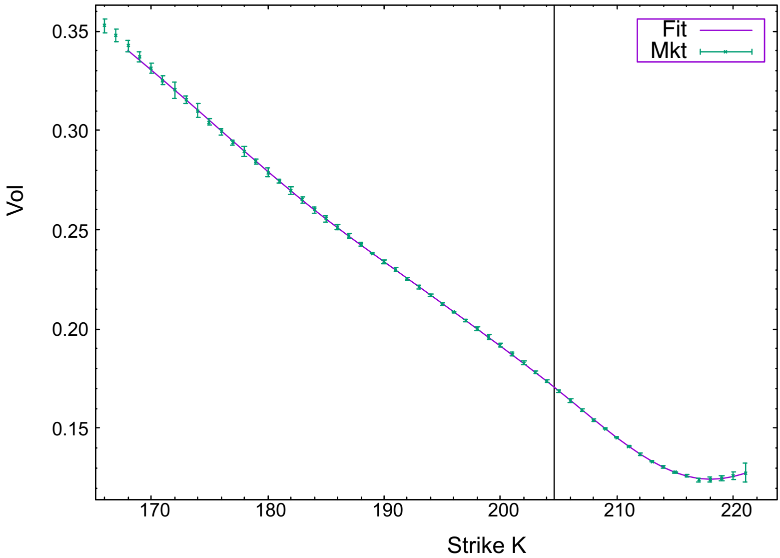
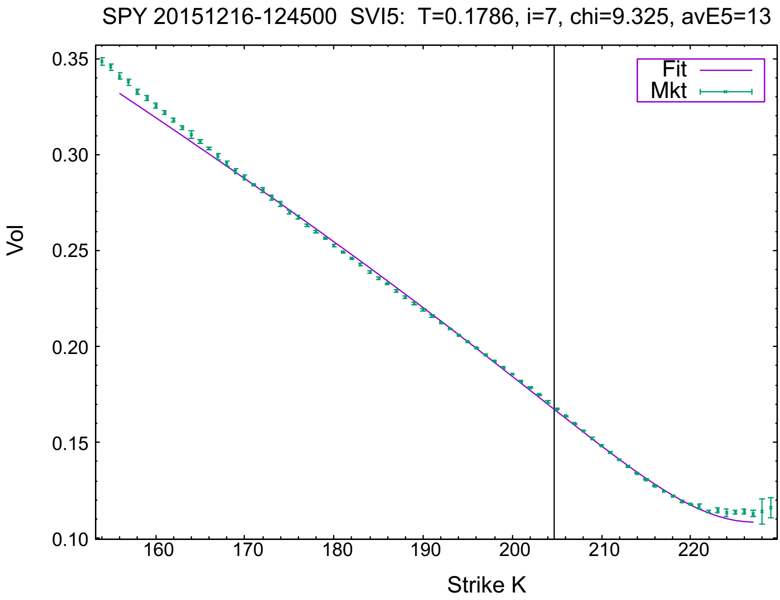
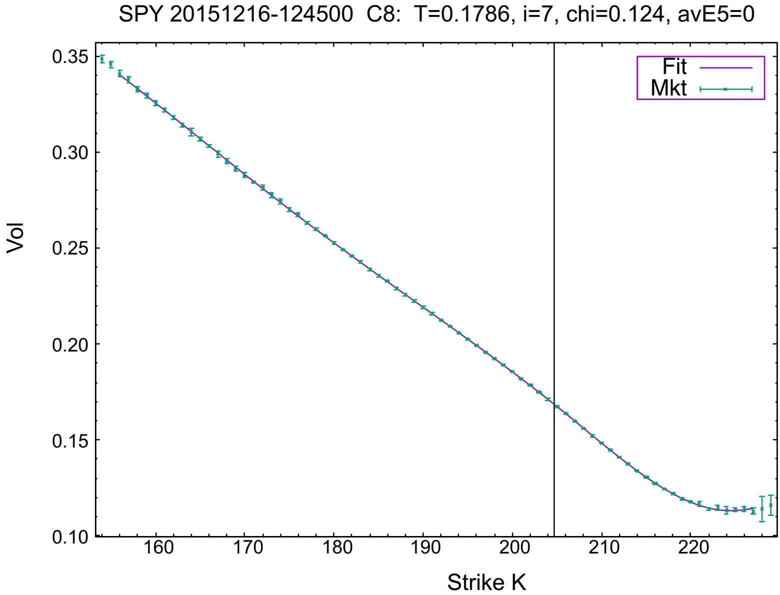
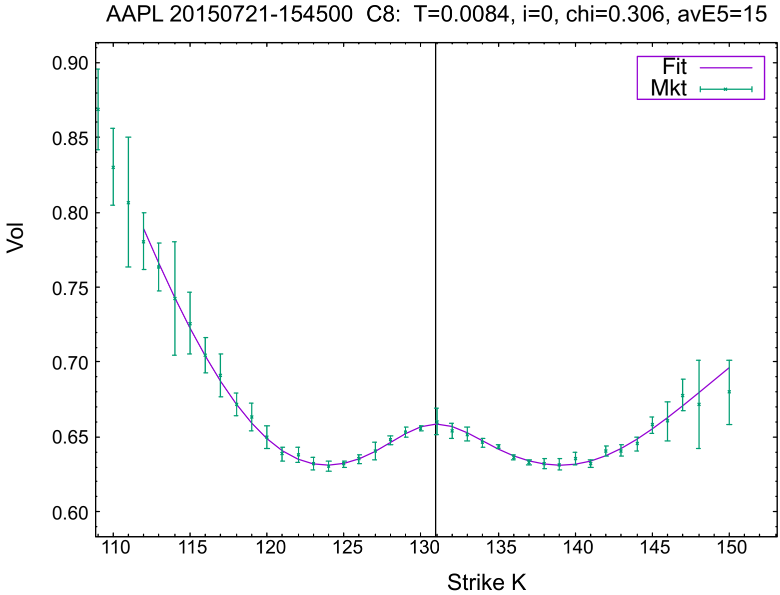
220
Strike K

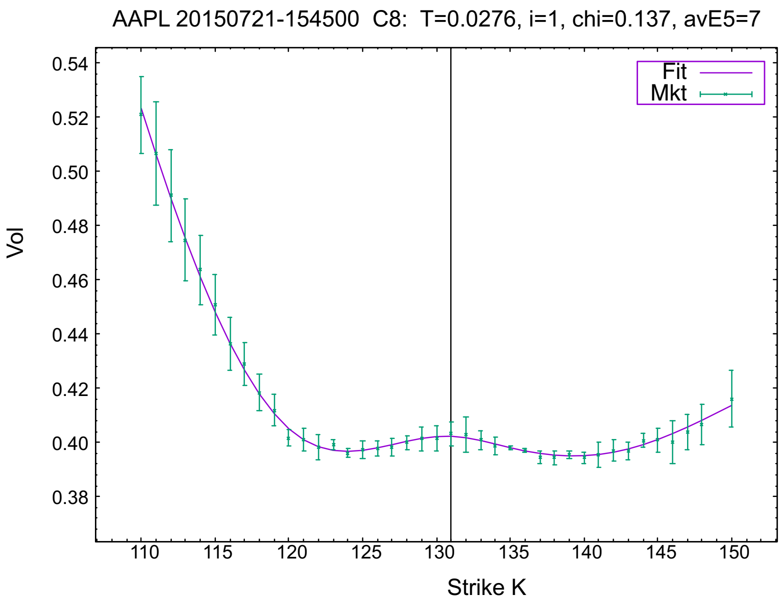
SPY 20151216-124500 C8: T=0.0827, i=4, chi=0.109, avE5=1



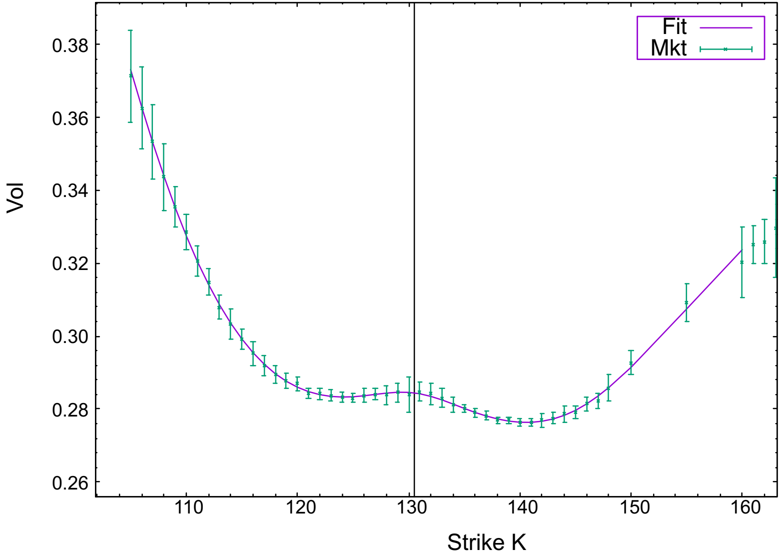

SPY 20151216-124500 SVI5: T=0.1786, i=7, chi=9.325, avE5=13
Fit
Mkt
Vol
0.35
0.30
0.25
0.20
0.15
0.10
160
170
180
190
200
210
220
Strike K

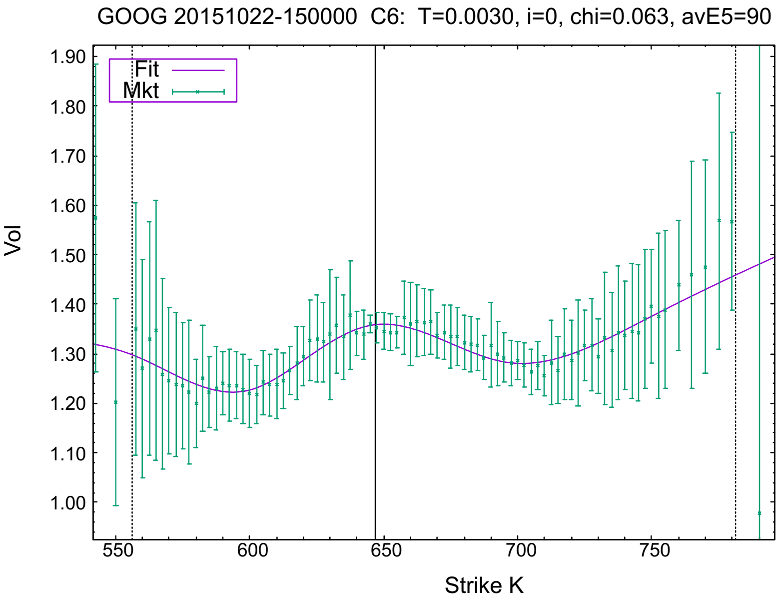

SPY 20151216-124500 C8: T=0.1786, i=7, chi=0.124, avE5=0
Fit
Mkt
Vol
0.35
0.30
0.25
0.20
0.15
0.10
160
170
180
190
200
210
220
Strike K

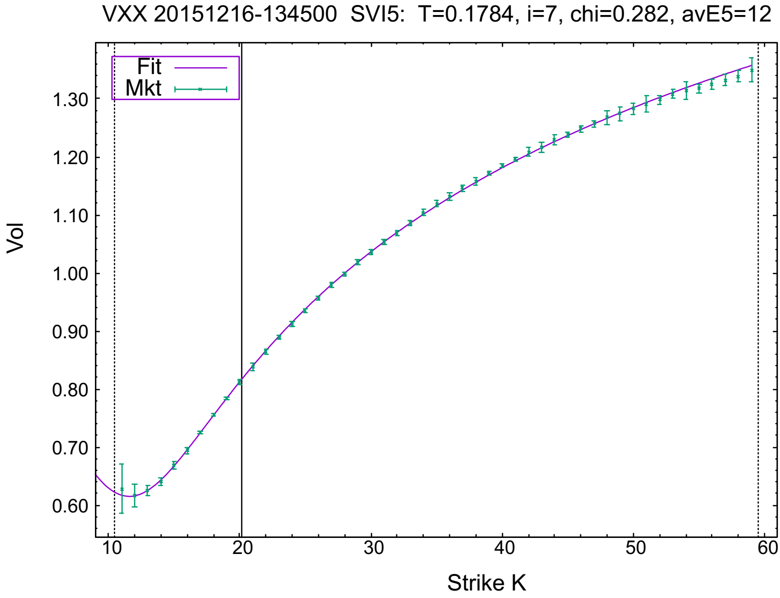

AAPL 20150721-154500 C8: T=0.0084, i=0, chi=0.306, avE5=15
Fit
Mkt
Vol
0.90
0.85
0.80
0.75
0.70
0.65
0.60
110
115
120
125
130
135
140
145
150
Strike K

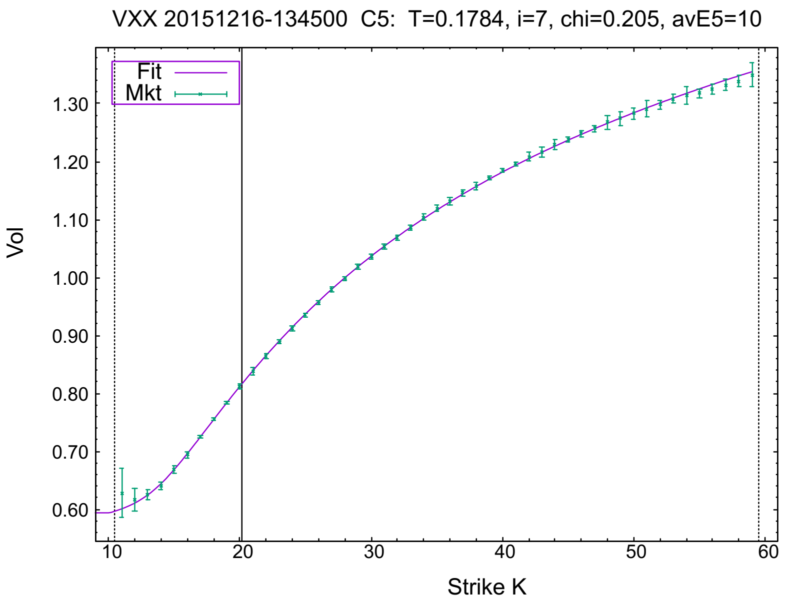

AAPL 20150721-154500 C8: T=0.0276, i=1, chi=0.137, avE5=7
Fit
Mkt
Vol
0.54
0.52
0.50
0.48
0.46
0.44
0.42
0.40
0.38
110
115
120
125
130
135
140
145
150
Strike K

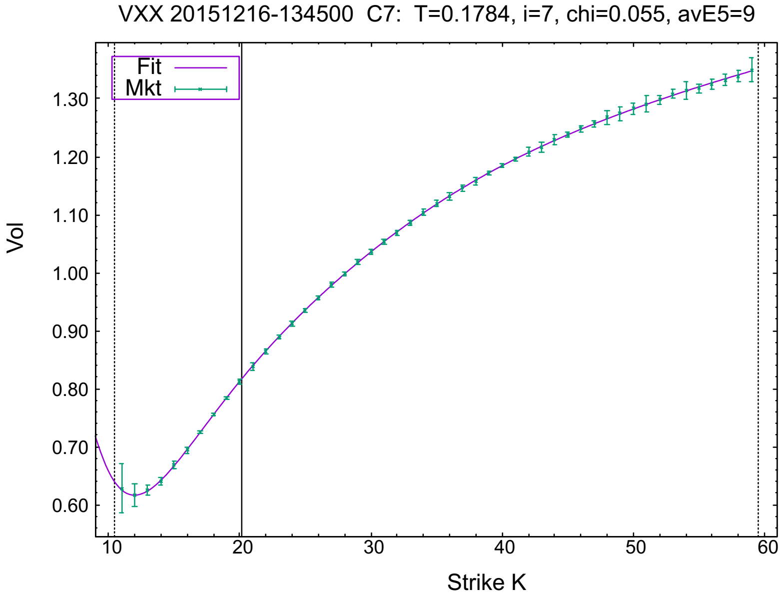
AAPL 20150721-154500 C8: T=0.0851, i=4, chi=0.055, avE5=5

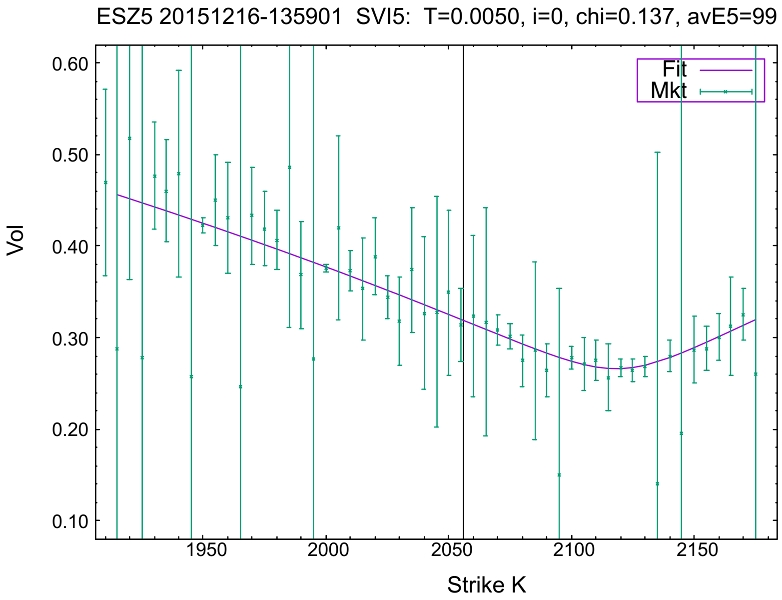


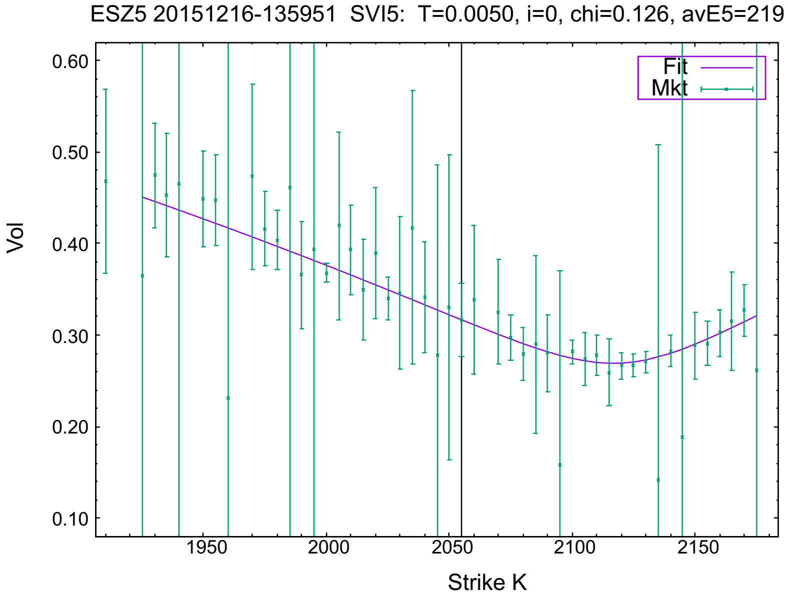

GOOG 20151022-150000 C6: T=0.0030, i=0, chi=0.063, avE5=90
Fit
Mkt
Vol
1.90
1.80
1.70
1.60
1.50
1.40
1.30
1.20
1.10
1.00
550
600
650
700
750
Strike K


VXX 20151216-134500 SVI5: T=0.1784, i=7, chi=0.282, avE5=12
Fit
Mkt
Vol
1.30
1.20
1.10
1.00
0.90
0.80
0.70
0.60
10
20
30
40
50
60
Strike K


VXX 20151216-134500 C5: T=0.1784, i=7, chi=0.205, avE5=10
Fit
Mkt
Vol
1.30
1.20
1.10
1.00
0.90
0.80
0.70
0.60
10
20
30
40
50
60
Strike K


VXX 20151216-134500 C7: T=0.1784, i=7, chi=0.055, avE5=9
Fit
Mkt
Vol
1.30
1.20
1.10
1.00
0.90
0.80
0.70
0.60
10
20
30
40
50
60
Strike K


ESZ5 20151216-135901 SVI5: T=0.0050, i=0, chi=0.137, avE5=99
Fit
Mkt
Vol
0.60
0.50
0.40
0.30
0.20
0.10
1950
2000
2050
2100
2150
Strike K


ESZ5 20151216-135951 SVI5: T=0.0050, i=0, chi=0.126, avE5=219
Fit
Mkt
Vol
0.60
0.50
0.40
0.30
0.20
0.10
1950
2000
2050
2100
2150
Strike K

Summary and Conclusion

- There is no standardization in the equity options markets around dividend modeling, borrow costs, or vol curves and their calibration.
- No borrow or vol curves are publicly available, historical or live, free or for purchase!
- No vol curves in the public domain can fit liquid names like SPY and AAPL. (Some believe they can only be fit non-parametrically...)
- This lack of transparency hinders the wider use of options and the efficient transfer of vol information across related products.
- Equity options are due for some major "RND": Rationalization, Normalization and Democratization!
- Hopefully can achieve same as in transition from old to new VIX: A healthier market, larger volumes, esp. from smaller players.
- Want to help? Stay tuned!

VOLAR – What we do

- Super-fast, robust, and sensible pricing, fitting, and vol curve analytics
  - Drop-in replacements for pain points in most firms infrastructure: pricer and fitter (simple API, hard analytics underneath)
- Provide, in real-time and historical, all valuation, risk, and trade analysis data and services relevant for options trading firms.
- Consulting, custom design and development
- For more information: `info@volar.xyz`