Equity Implied Vols for All, Part 1: Pricing with Cash Dividends

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Outline

- Introduction
- Dividend Modeling
- Hybrid Models
- Calibration
- Arbitrage
- Conclusion

Details and references in:

Pricing Vanilla Options with Cash Dividends (at ssrn)

Why Dividend Modeling?

- Many stocks and ETFs (incl. SPY, QQQ, IWM) in the US, at least, pay cash dividends.
- Their options are American exercise-style (except index options).
- 40y after Black-Scholes-Merton there is no agreement on how to model dividends! Different ways of handling them can give quite different implied vols (even qualitatively: term-structure, skew).
- The modeling question is: What should replace geometric Brownian motion (GBM) in the presence of cash divs?
- Even for vanillas: This matters not just for American but also for European exercise-style options (even after calibration...). And: How to compare e.g. SPY and SPX vols?
- Vanillas are usually modeled such that *something* follows GBM; exotics require more fancy "LSVJ" etc models.
  But the first two modeling/calibration issues are the same:
  What borrow costs and dividend model to use?

How Does Dividend Modeling Matter?

- Most obviously, exotics calibrated with different dividend models to the same vanilla market have different prices and greeks (see literature on variance swaps, barriers, etc).
- Also obvious: For vanillas different dividend models will give different greeks, even if they agree on prices.
- Less discussed are these questions: To what extent can different dividend models match a full set of vanilla market prices? What, if any, price differences remain *after calibrating* different dividend models to the same vanilla market? How do their calibrated parameters differ? What does "the market" actually do?
- If possible, dividend modeling should capture all relevant features, but be simple enough to allow consistent modeling of vanillas *and* exotics.

What's Hard about Cash Dividend Modeling?

- Cash dividends mean that the observed stock price can not follow GBM.
- One hope: Find *some* transformed quantity that can still follow GBM while providing an economically sensible model for cash dividends.
- Alternatively, give up on GBM and assume the observed stock price follows GBM between ex-div dates, and has jumps by the dividend amounts at the ex-div dates.
    - Known as *spot model* or *piece-wise log-normal model*.
    - This might have the most popular model among OMMs at some point, perhaps not anymore.

What do Options Market Makers do?

- Mostly take p.o.v. of listed vanilla options market maker (OMM): SIG, Citadel, IMC, Getco/KCG, some banks, etc.
  (Compare to OTC and (light) exotics MM, EqDeriv Desks).
- Most provide and take liquidity. Need good valuation at all times.
- Work in "hacked" Black-Scholes framework for valuation:
  - Pick dividend model, as well as dividend amount, dates.
  - Figure out borrow cost term-structure.
  - Use a *vol-time* aka *trading-time* (affects relative EE premia).
  - Decide on event weights (underlier specific: FOMC, earnings).
  - Manage/fit implied volatility curves (surface).
  - Have good and fast underlier valuation.
- Questions: Why Black-Scholes? Should there be one borrow per term? Should same implied vols be used for call and put at a given maturity and strike T, K? Should we price with rate and/or vol term-structure?

What are the hard quant problems OMMs face?

Whether you're just trying to match the market or trying to *set/disagree* with the market, you need to be able translate market prices into the Black-Scholes language OMMs think in:

- Which cash dividend model to use?
- How to imply borrow cost term-structure?
- How to design sensible implied vol curves/surfaces?
- How to fit vol curves/surfaces, robustly and fast?
- Vol dynamics: How does vol surface move move spot moves, etc.
- And many many more questions related to quote logic, risk management, details of how to deal with "crazy" markets, etc.
- Some are chicken-and-egg questions: What does the market *do*?

What do we mean by a Dividend Model?

- Mathematically: What is the SDE of the underlier?
- In practice: How to combine the stochastic part of underlier evolution (e.g. *who* follows GBM?) with...
- Three types of dividends:
  - A dividend yield – used to model borrow cost
  - Cash dividends – how most dividends are actually paid
  - Discrete proportional dividends
- Most firms use *blending scheme* to transition from cash dividends on short end to proportional dividends in long term[1] – this is the easy modeling part; various calibration approaches possible.
- Proportional divs are also useful in times of extreme uncertainty (market-wise or name-specific). E.g. during 2008 crisis.

[1] For forward-starting options on dividends can't use static blending scheme: conditional on future spot, dividend volatility would be too large.

Notation, the Forward $F_T := E[S_T]$

- Let r_t, q_t denote (deterministic) discount and borrow rates, and $\mu_t := r_t - q_t$ the drift, with time $t = 0$ being "now".
- Let δ_i, d_i denote the proportional and cash dividends expected to be paid at ex-dividend dates t_i.
- If the forward jumps across a dividend date $F_{t_i^+} = (1 - \delta_i) F_{t_i^-} - d_i$ then

$$F_T = f_p(T) \left(S_0 - \sum_{i:t_i \leq T} \frac{d_i}{f_p(t_i)} \right)$$

where the *proportional growth factor* is:

$$f_p(t) := \exp\left(\int_0^t \mu_t dt\right) \prod_{i:t_i \leq t} (1 - \delta_i) .$$

The Spot Model (SM)

- Recall: GBM between div dates, jumps at ex-div dates.
- It can happen that spot drops below next dividend – need *dividend policy* to specify what happens.
- Does NOT have above formula for forward, nor analytic European prices; they depend on dividend policy.
- Solving by FD (even worse: Tree) methods requires more effort for reasonable accuracy (relative to...)
- Does not easily integrate with (light) exotics modeling.
- Nevertheless, sometimes (used to be?) considered to be gold-standard of vanilla cash dividend modeling.

The (full) Hybrid Model (FHM), Buehler et al

- Alternative view: Imagine the stock/ETF to be composed of a deterministic cash-dividend related piece, and a fluctuating remainder:

$$S_t \;=\; (F_t - D_t)X_t \;+\; D_t \;, \qquad E[X_t] = 1 \quad \text{for all } t$$

where the shift D_t is the PV of all divs after t

$$\text{FHM:} \qquad D_t \;:=\; f_p(t) \sum_{i:t<t_i<\infty} \frac{d_i}{f_p(t_i)}$$

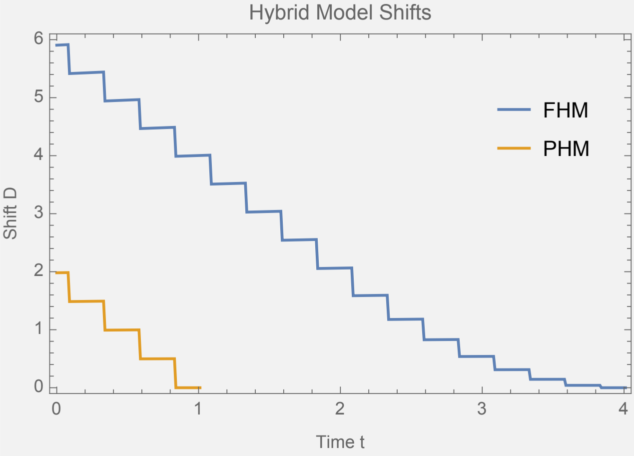
- For consistency we have to consider *all* cash dividends (before the end of the blending transition to purely proportional dividends) .
- Is perfectly consistent model, easy to integrate with exotics.
- Pricing depends on all dividends, incl. dividends after maturity!?
- If you really worry about what happens at very low spot, consider using more proportional dividends or default modeling!

The Partial Hybrid Model (aka Escrowed Dividend Model)

- Modify (full) hybrid model so that shift D_t only involves dividends up to maturity T:

$$\text{PHM}: \qquad D_t := f_p(t) \sum_{i:t<t_i \leq T} \frac{d_i}{f_p(t_i)}$$

- Now pricing does not depend on dividends after maturity.
- But we have a different SDE when T goes through an ex-div date!
- Obviously leads to arbitrage – even ATM – unless implied volatility jumps up suitably when going through ex-div date!
- Since $D_T = 0$, the implied vols (in Euro case) are just Black vols.
- Is/was commonly used; goes back to (at least) J. Hull's textbook, and Roll, Geske, Whaley for the one-dividend case.



The shifts D_t for the full and partial hybrid models with $r = 3\%$, $q = 1\%$, and a quarterly cash dividend of 0.5 first paid at $t_1 = 0.085$, using blending scheme $(2, 4)$y.

For PHM we use $T = 1.01$.

The SKA and BV (Bos-Vandermark) Models

- Bos, Vandermark proposed an approximate solution to the SM for European vanillas. It is popular for (European) index options.
- It fits into shift framework, and has a simple motivation: In SM close divs act like spot-adjustment, far divs like strike-adjustment.
- Recipe for Euro options: $S_0 \to S_0 - \hat{D}^{(n)}$, $K \to K + f_p(T)\hat{D}^{(f)}$

$$\hat{D}^{(n)} := \sum_{i:0<t_i\leq T} \left(1 - \frac{t_i}{T}\right) \frac{d_i}{f_p(t_i)} \,, \quad \hat{D}^{(f)} := \sum_{i:0<t_i\leq T} \frac{t_i}{T} \frac{d_i}{f_p(t_i)}$$

- Two variations exist to convert this into full-fledged stochastic process that can be used for American options. SKA variant:

$$\text{SKA: } D_t \; := \; f_p(t) \sum_{i:t<t_i\leq T} \frac{d_i}{f_p(t_i)} - f_p(t)\hat{D}^{(f)} \; = \; D_t^{(PHM)} - f_p(t)\hat{D}^{(f)}$$

$$F_t - D_t \; = \; f_p(t)\left(S_0 - \hat{D}^{(n)}\right) \; = \; f_p(t)\tilde{S}_0$$

- The SKA process SDE has a continuous dependence on T.

The SKA and BV Models (cont'd)

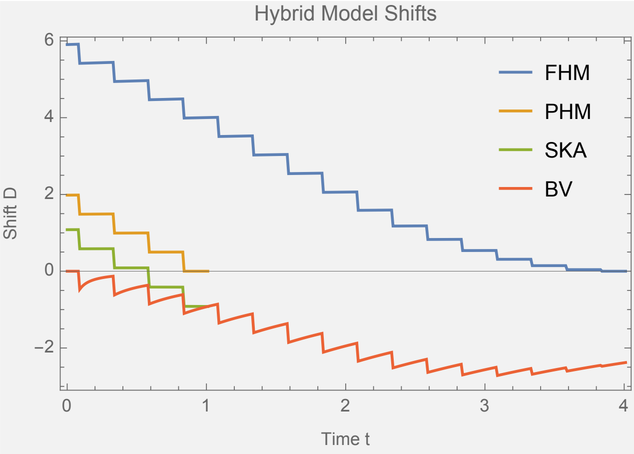
- The alternative extension to the American case, denoted as BV (the literal interpretation?), is defined via shift

$$\text{BV}: \qquad D_t := -f_p(t) \sum_{i:0<t_i\leq t} \frac{t_i}{t} \frac{d_i}{f_p(t_i)}$$

- This is a maturity-independent shift, but now there is an extra time- and div-dependent drift term in $S_t = (F_t - D_t)X_t + D_t$:

$$F_t - D_t = f_p(t)\left(S_0 - \sum_{i:0<t_i\leq t}\left(1 - \frac{t_i}{t}\right)\frac{d_i}{f_p(t_i)}\right) \neq f_p t(t)\,\tilde{S}_0$$

- Not clear if SKA or BV have been used in American case before.
- SKA and BV both allow $S_t < 0$, since $D_t < 0$ for some t.
- We will concentrate on FHM, PHM, SKA.



The shifts D_t for various hybrid models with $r = 3\%$, $q = 1\%$, and a quarterly cash dividend of 0.5 first paid at $t_1 = 0.085$, using blending scheme $(2, 4)$.

For PHM, SKA: using $T = 1.01$. BV has dividend- and t-dependent drift.

Dividend Modeling Summary and Issues

- Besides the spot model, all other (sensible) proposals fall into the hybrid model framework, i.e. correspond to suitable dividend-dependent shifted GBM. (Except for P. Jaeckel's finite state space transition density framework DHI.)
- Dividend modeling is an under-studied problem; there certainly is no consensus. There is some confusion in the literature, as well as among SW and options data vendors, about the various proposed models, approximations, and numerical algorithms.
  There are not even standard names for the proposed models.
- Many approximations and recipes used in the past are not generic: Some are specific to European options, and/or cover only the case of one dividend (or too slow for more than 1 or 2), and/or only work for calls but not puts, and/or assume borrow cost $q \leq 0$.
- Borrow costs are also a widely under-appreciated subject.

Dividend Modeling: Wish List

- Modeling and practical implementation should be generic:
  - Cover all US vanilla options on stock, ETFs, futures, indices.
  - Euro or American. Cover global idiosynchracies (India, Brazil).
  - Any number of dividends (e.g. DIA has them monthly).
  - Handle large cash dividends.
  - Allow large borrow costs (IWM, leveraged ETFs, recently IPO'd names, times of crisis, etc).
  - Ideally have analytic forwards and European prices.
  - Should be easy to handle business time, events, default risk.
  - Preferrably extendable to (at least, light) exotics.
- Simple; economically sensible motivation.
- No arbitrage (at least negligible in practice compared to spreads).
- Fast in practice: In US alone there are 600,000+ options trading.

The Hybrid Model (HM) Framework

- Hybrid models provide an appealing framework for vanilla and exotic modeling with cash dividends.
- Cash dividends produce no extra complications compared to the LSVJ-default modeling one already has to do.
- For Europeans, the un-discounted price of a vanilla with a shift D_T at maturity T, strike K is, in terms of the Black-formula

$$\hat{V}(F_T - D_T, K - D_T, \sigma\sqrt{T}) \ .$$

- All hybrid models have a shift function D_t,

$$S_t \ = \ \tilde{S}_t + D_t \ = \ (F_t - D_t)X_t + D_t \ ,$$

  where for the *pure stock* X_t (or $\tilde{S}_t$) we assume GBM for vanillas.
- To price (American) vanillas, we can use our favorite numerical BS pricer for $\tilde{S}$ with a time-dependent adjusted strike $\tilde{K}_t := K - D_t$.
- Are numerically more benign than SM.

Hybrid and Spot Models Compared

Feature	HM0	HM1=PHM	HM2=FHM	HM3=SKA	SM
SDE	GBM	shiftedGBM	shiftedGBM	shiftedGBM	pwGBM
Need small S dividend policy	No	No	No	No	Yes
Consistent SDE across terms	Yes	No	Yes	Almost	Yes
Arbitrage (accu implementation)	No	Yes	No	Little	No
Usual, exact forward formula	Yes	Yes	Yes	Yes	No
Exact, fast Euro prices	Yes	Yes	Yes	Yes	No
Pricing depends on divs $> T$	No	No	Yes	No	No
Spot adj at T: $D_0(T)$	0	> 0	> 0	> 0	0
Strike adj at T: $D_{t=T}(T)$	0	0	≥ 0	< 0	0
Spot adj $D_0(T)$ thru t_d	0	Jump up	Constant	Continuous	0
Strike adj $D_{t=T}(T)$ thru t_d	0	0	Jump down	Jump down	0

Comparison of hybrid models and the spot model in the presence of cash dividends.

GBM might include proportional jumps from discrete proportional dividends.

HM0 is a degenerate version of a HM where all cash dividends have been converted to proportional dividends.

Hybrid Model Relationships

- Hybrid models can have very different financial properties. Are there general relationships? If one can match the market, can all?
- Consider the calibration problem: Given some market call and put prices C, P at given T, K. How are the calibrated borrow rate q and vol σ in different HMs related? Look at a surface of T, K.
- All HM have the same exact formula for the forward, so in the Euro case all implied borrow rates q are the same! (Not true with HM0 or SM).
- Different HM implied vols in the Euro case are analytically related (via inversion of Black formula...). Simple high-accuracy formulas exist for ATF vols.
- What about the American-exercise case?

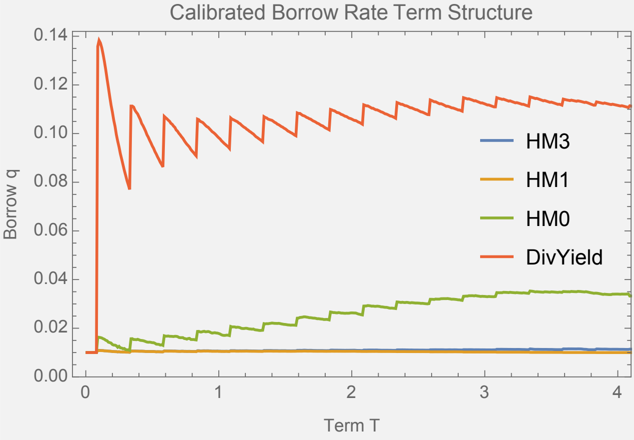
Numerical Implementation

- Do we need finite-difference (FD) methods for accurate vanilla pricing with discrete dividends?
- Maybe for SM (no study...). But for HMs cash dividends are taken largely out of pricing problem; appear as time-dependent strike.
- For HMs finely-tuned tree methods can achieve the kind of accuracy and smoothness needed in practice.
- Will use Leisen-Reimer (LR) trees:
  - Have 2nd order convergence in # tree steps N in Euro case.
  - Some studies (w/o discrete divs) indicate they are also more efficient than CRR etc trees & FD schemes in American case.
  - Can be adapted to discrete dividends, with various tricks.
- Checked against "all" results in literature, and the few exact results for American case (1d-integral for American call with 1-div, $q \leq 0$).
- Speed/price: 12 μs on $N = 101$ step tree, 3.5GHz Intel Core i5.

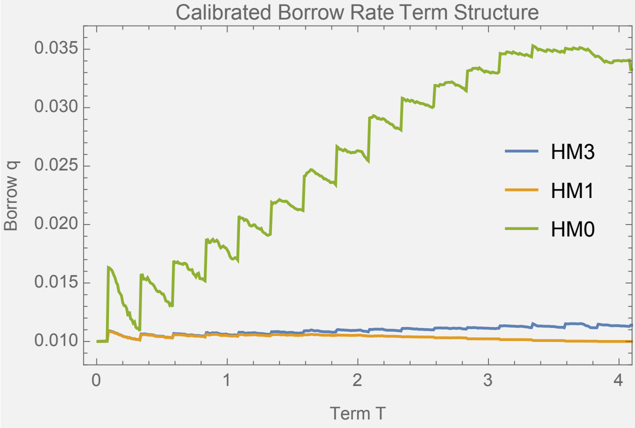
Calibrating Hybrid Models to "The Market"

- As reference "market" choose e.g. Full Hybrid Model HM2, which is arbitrage-free. Use American exercise (realistic & interesting case).
- Now calibrate HM1, HM3 (and HM0) against it: For each T, K find q, σ that match[2] the American call and put in the "market".
- For HM0 choose $\delta_i = d_i / S_0$.
- Choose pretty "hard" case: HM2 with $S_0 = 100$, $\sigma = 30\%$, $r = 3\%$, $q = 1\%$, $\delta_i = 0$, quarterly cash divs, usually $d_i = 2$, with $t_1 = 0.085$. Means: cash div yield of ca. 8%, much "harder" than SPY's 2%.
- Blending scheme $(2, 4)$y.
- How do implied borrow and vol term-structures and skews look?

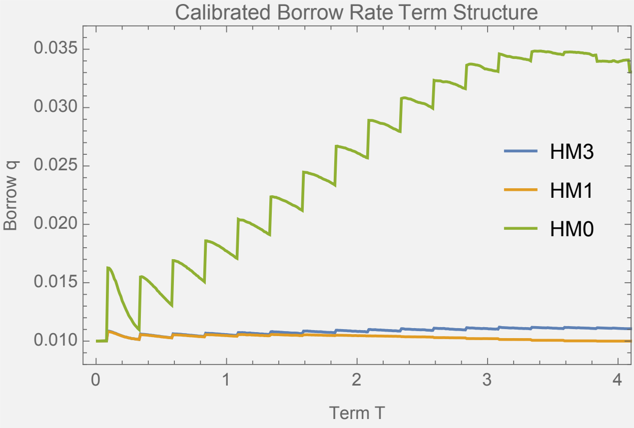
[2] The calibration algorithm is accurate to $\mathcal{O}(10^{-7})$ or better.



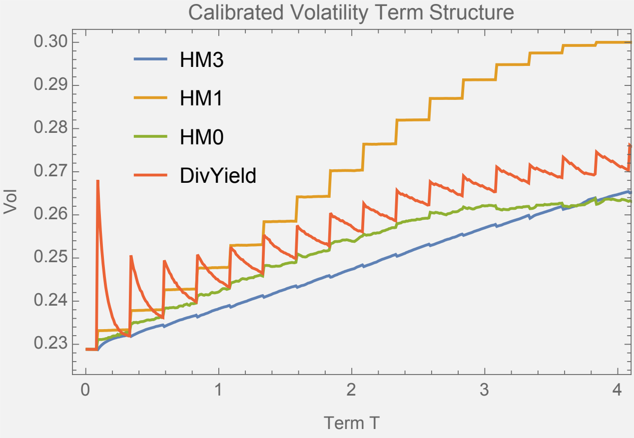
Implied ATF borrow rate term-structure of various models calibrated to reference market HM2 with $r = 3\%$, $q = 1\%$, $\sigma = 30\%$, and a quarterly cash dividend of 2 first paid at $t_1 = 0.085$. Exercise-style is American and $N = 65$.



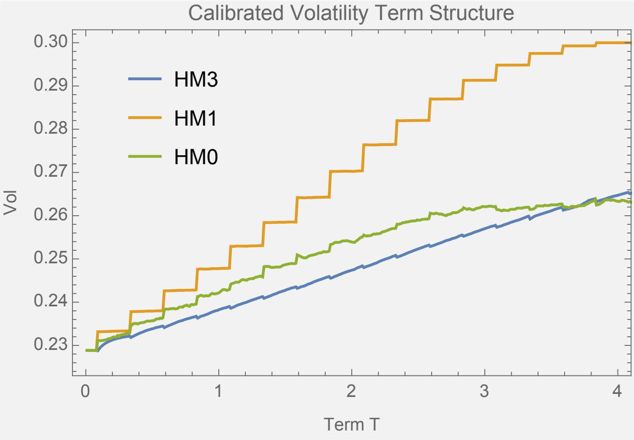
As above, for just HM0, HM1, HM3.



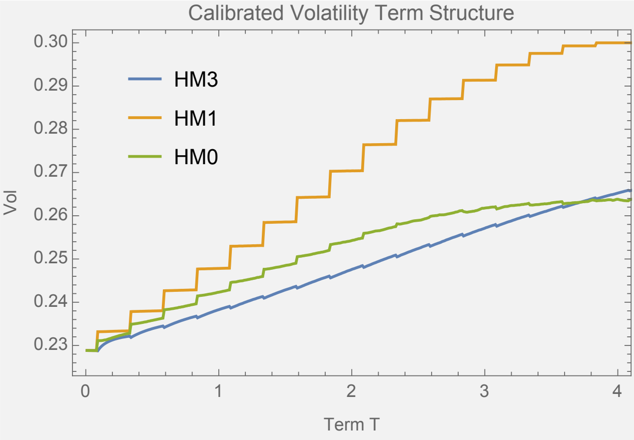
As above, for $N = 651$.



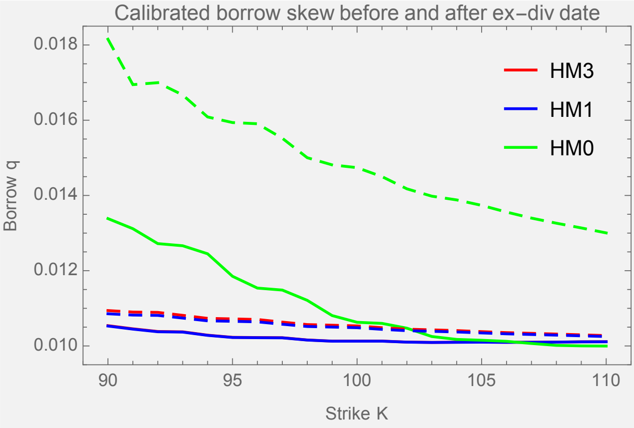
Implied ATF volatility term-structure of various models calibrated to reference market HM2 with $r = 3\%$, $q = 1\%$, $\sigma = 30\%$, and a quarterly cash dividend of 2 first paid at $t_1 = 0.085$. Exercise-style is American and $N = 65$.



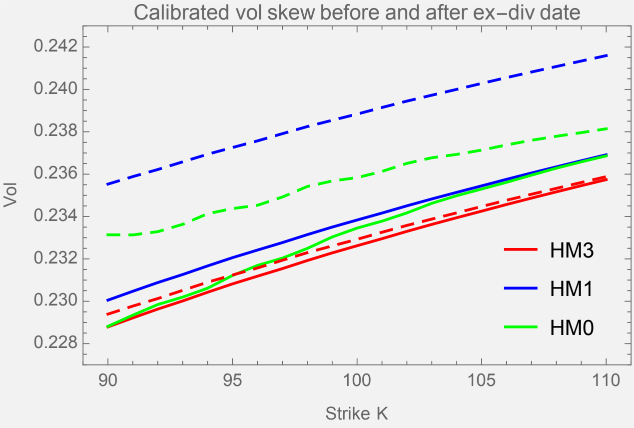
As above, for just HM0, HM1, HM3.



As above, for $N = 651$.



Implied borrow skew of various models calibrated to reference market HM2 with $r = 3\%, q = 1\%, \sigma = 30\%$, and a quarterly cash dividend of 2 first paid at $t_1 = 0.085$. We show the skew just before, $T = 0.33$ (solid), and after, $T = 0.34$ (dashed), the second dividend. Exercise-style is American and $N = 65$.



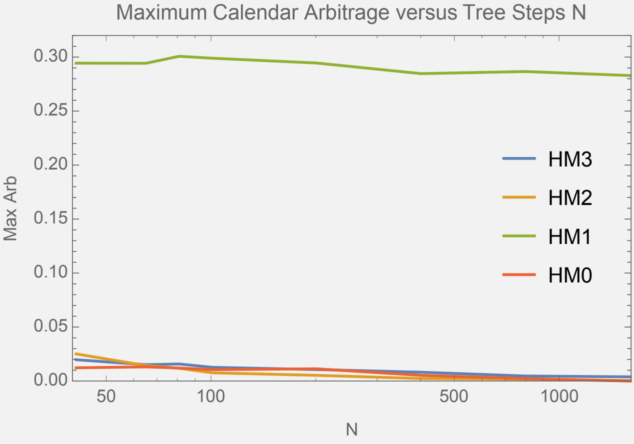
As above, for the implied volatility skew.

Hybrid Model Calibration Results: Conclusions

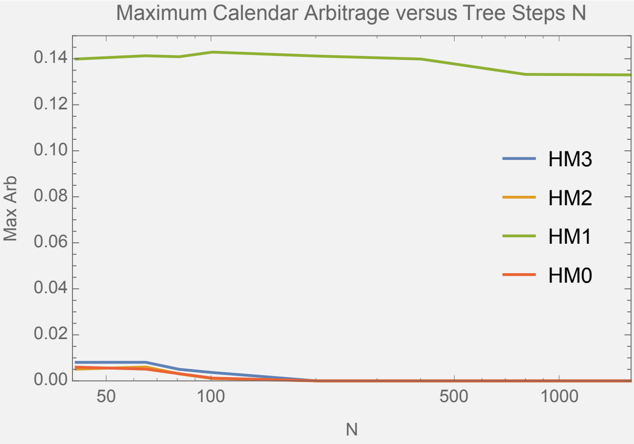
- If one HM can be calibrated to the market, so can any other.
    - HM1 implied volatilities jump up across ex-dividend dates.
    - For $K \leq \mathcal{O}(D)$ we can have $P \equiv 0$ unless default allowed.
- Implied borrows are exactly the same in Euro case for all T, K, and extremely close in American case (SPY-like: < 1bps for $T \leq 4$y).
- The implied vol relationships between HMs defined via their resp. Black-formulas is exact in European case for any $T, K > \max(D)$.
- Same relationships holds to *extremely* high accuracy also in American case (SPY-like: $\ll 1$bps for $T \leq 4$y).
- These relationships between HM1, HM2, HM3 are essentially exact for practical purposes, even when early-exercise premia are large, even for small N!
- They are least 100x more accurate than other relationships between dividend models discussed in the literature. Why?

Calendar Arbitrage

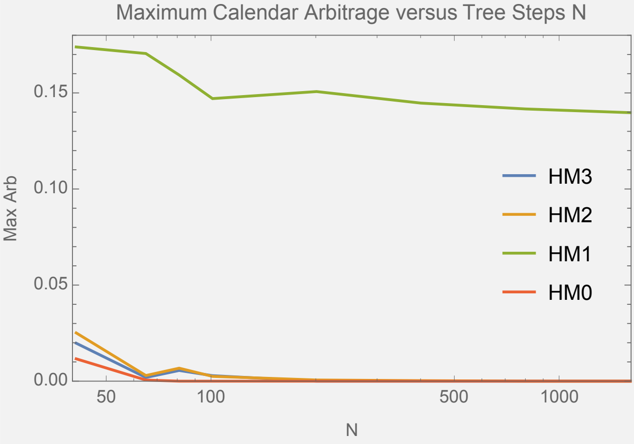
- Arbitrage arising from "bad" volatility surfaces is an important separate subject to be discussed elsewhere.
- Here we just consider calendar arbitrage for a flat vol surface. Then the question is whether the prices of American call options at fixed strike can decrease as T increases.
- Consider all hybrid models with flat rate and volatility inputs. With infinite accuracy HM2 (and HM0) do not have arbitrage; HM1 does. What about finite N? What about HM3?
- Find maximum calendar arbitrage by sampling wide range of strikes and finely spaced terms T; plot as function of tree steps N.



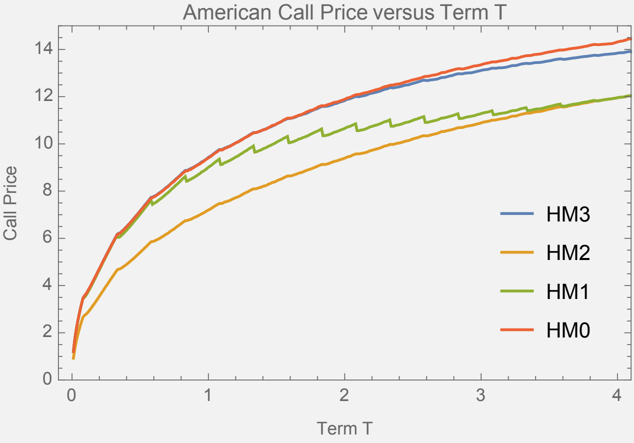
Maximum calendar arbitrage for American calls as a function of the number of steps in our tree, in various models, with $r = 3\%, q = 1\%, \sigma = 30\%$, and a quarterly cash dividend of 2 first paid at $t_1 = 0.085$. The max is calculated over $K = 80 \ldots 100$, with T sampled every 0.01y up to $T = 4.10$y.



As above, with a dividend of $d = 1$.



As above, with a dividend of $d = 2$, but sampled up to $T = 0.60$y



American call price with strike $K = 100$ as a function of time to maturity T in various models, with $r = 3\%, q = 1\%, \sigma = 30\%$, and a quarterly cash dividend of 2 first paid at $t_1 = 0.085$. The number of steps in our tree is $N = 65$.

Conclusion

- There is no market standard for pricing (even) vanilla options with cash dividends.
- Besides SM, all of HM1, HM2, HM3 (only Euro?) seem to be in use, with varying qualities of implementations. History obscure.
- We believe that market participants should converge on Hybrid Models to price with cash dividends:
  - Are simple, generic, extendible (credit, exotics, etc).
  - Can all be implemented via very fast tuned LR trees.
- The exact relationships of the European HM vols carry over to high accuracy to the American case, even on small pricing grids.
- Vol surface management is much simpler with HM2, HM3.

Outlook

- After dividend modeling... even less standardization exists around borrow costs, vol curves, and their calibration.
- No borrow or vol curves are publicly available, historical or live, free or for purchase!
- This lack of transparency and consistency hinders the wider use of options and the efficient transfer of vol information across related products.
- Equity options are due for some major "RND" – Rationalization, Normalization and Democratization!
- Hopefully can achieve same effect as in transition from old to new VIX: A healthy market, larger volumes, esp. from smaller players.
- Want to help? Stay tuned!