

High Performance Options Analytics and Volatility Modeling

Timothy R. Klassen

Founder and CEO, Volar Technologies LLC

timothy.klassen@volar.io

Columbia University

April 13, 2016

Mysteries of the (Listed) Equity Options World

- Why do data vendors have different vols for calls and puts, no greeks for many options (ITM, some OTM), and bad implied dividends?
- Why do CNBC, Bloomberg, et al never show sexy vol curves?
- Why does every half-way serious options trading team write their own valuation (pricer, fitter for borrows and vols) and trade analysis infrastructure (mark-ups, PnL decomposition, TCA, etc)?
- Lead-lag relationships in vol space are orders of magnitude slower than in the equity domain.
- Some leading options market makers (OMMs) use hand-calibrated (not auto-fitted) vol surfaces.

Equity Options Market Overview

- In US alone there are 500,000+ options on 4,000+ underliers.
 - Most do not trade on any given day. Most have very wide bid-ask spreads at any given time, especially ITM and very OTM options.
 - Even liquid underliers can have such options.
 - OTOH: Some options have super-tight spreads that are hard to fit with “reasonable” vol skew curves.
- All options can only be valued with real-time, robust implied borrow and sophisticated volatility curves.
- Also required for real-time risk and PnL decomposition.
- Well-designed parametric curves are needed for sensible book-level sensitivities (vanillas + exotics): *normalized vega*, *skew vega*, etc
- All borrow and vol curves are proprietary. Despite big efforts, no data/analytics vendor has them (even EOD historical).

Mysteries.... in more detail

- The equity options market is complicated due to dividends, borrow costs, events, and vol curves with lots of structure – but spreads can be very tight. Nowadays most liquid options (SPY, QQQ, AAPL, VXX) are American-style (much harder than future/index options)
- To value all vanilla options on an equity underlier one needs:
 - Interest rates – freely available (but which one?)
 - Dividend projections – can be bought (but pretty expensive)
 - Borrow curve – not available for purchase at any price
 - Volatility surface, volatility TTX – not available at any price
- To value exotics, one needs an arbitrage-free volatility surface also in the far wings, as input to SLVJ calibrators (not available).
- A dirty secret of the options industry is: Only a handful out of hundreds of players know (sort of):
 - How to properly price with cash dividends (model & algo issue)
 - How to imply stable borrow cost curves
 - How to design, and robustly calibrate tradable vol curves in real-time

What do Vanilla Options Market Makers do?

- Use “hacked” Black-Scholes framework for valuation & risk mgmt:
 - Pick dividend model, as well as dividend amount, dates.
 - Figure out borrow cost term-structure.
 - Use a volatility TTX (affects relative early-exercise premia).
 - Decide on event weights (underlier specific: FOMC, earnings).
 - Manage or fit implied volatility curves/surface.
 - For greeks use implied vol $\sigma = \sigma(T, K)$ in Black-Scholes model, but correct for spot-vol dynamics (smart delta).
 - Have good and fast underlier valuation.
 - Need fast & robust American pricer (with proper div model).
- Why Black-Scholes?
- Should there be one borrow cost per term?
- Should same implied vol be used for call and put at a given maturity and strike T, K ?

Dividend Modeling

- Forty years after Black-Scholes there is no consensus on how to model cash dividends!
- Cash dividends mean that the observed stock price can not follow geometric Brownian motion (GBM).
- In a vanilla context the question is how to combine the stochastic part of underlier evolution (e.g. *who* follows GBM?) with...
- **Three types of dividends:**
 - A dividend yield – used to model borrow cost
 - Cash dividends – how most dividends are actually paid
 - Discrete proportional dividends
- Most firms use *blending scheme* to transition from cash dividends on short end to proportional dividends in long term.
- Proportional divs are also useful in times of extreme uncertainty (market-wise or name-specific). E.g. during 2008 crisis.

Dividend Models

- Main two classes of dividend models are:
 - *Spot model*: The dividends come out of the observed stock price. (Need to modify cash dividends at low stock price.)
 - *Hybrid models*: The dividends come out of a “cash buffer”, related to the PV of future dividends: $S_t = \tilde{S}_t + D_t$
- Spot model might seem naively more reasonable, but in practice leads to a lot of complications and hacks, since not GBM.
- Hybrid models are much simpler to handle for both vanillas and exotics, since *pure stock* \tilde{S}_t still follows GBM. Can also easily handle credit risk, extension to (light) exotics, local vols, etc.
- We will assume a hybrid model from now on.
- NOTE: Even if you care only about European options, dividend modeling matters – how e.g. are SPX and SPY vols related?

Hybrid Models, Notation

- In a hybrid model the stock follows *shifted GBM*, and the prices of (un-discounted) European vanillas for the pure stock are:

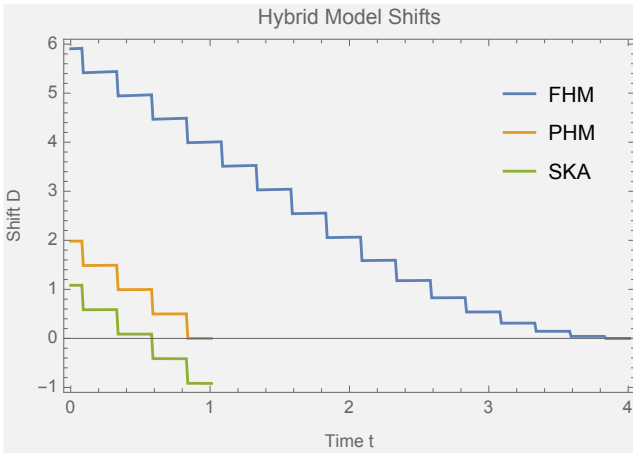
$$\hat{C} = + F N(d_+) - K N(d_-) \quad (1)$$

$$\hat{P} = - F N(-d_+) + K N(-d_-) \quad (2)$$

- Here $N(x)$ is the normal cdf, log-moneyness $y := \log(K/F)$, and

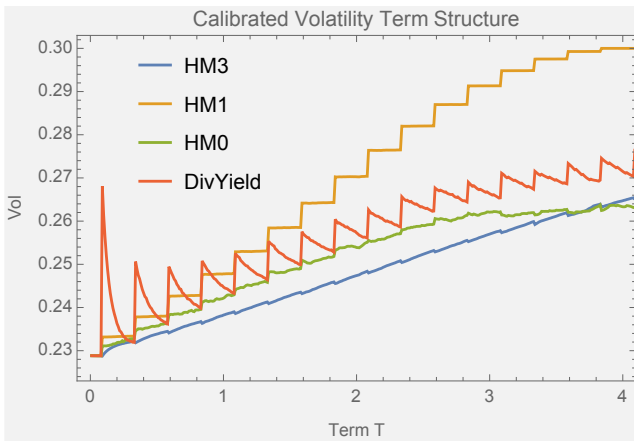
$$d_{\pm} := \frac{-y}{\hat{\sigma}} \pm \frac{1}{2} \hat{\sigma} \quad , \quad \hat{\sigma} := \sigma \sqrt{T}$$

- $\sigma = \sigma(T, K)$ is the implied volatility of the option.
- Normalized prices \hat{V}/F are function of two dim-less variables: $y, \hat{\sigma}$.
- Actual prices are obtained by shifting the forward $F = F_T$ and strike K by the shift D_T , that depends on the hybrid model.
- For details: *Pricing Vanilla Options with Cash Dividends* (SSRN).

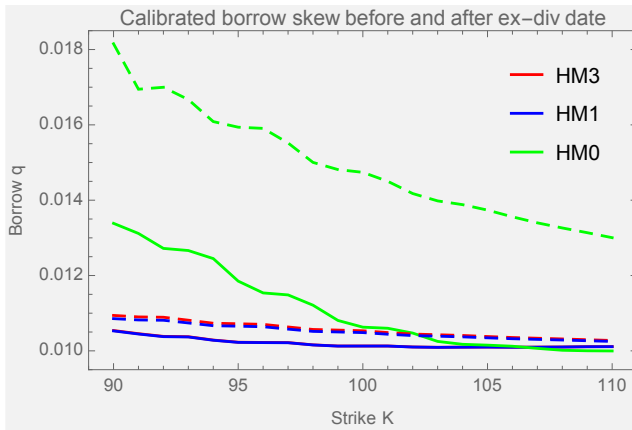


The shifts D_t for various hybrid models with $r=3\%$, $q=1\%$, and a quarterly cash dividend of 0.5 first paid at $t_1=0.085$, using blending scheme (2, 4).

For PHM, SKA: using $T=1.01$. FHM = HM2, PHM = HM1, SKA = HM3



Implied ATF volatility term-structure of various models calibrated to reference market HM2 with $r=3\%$, $q=1\%$, $\sigma=30\%$, and a quarterly cash dividend of 2 first paid at $t_1=0.085$. Exercise-style is American and $N=65$.



Implied borrow skew of various models calibrated to reference market HM2 with $r=3\%$, $q=1\%$, $\sigma=30\%$, and a quarterly cash dividend of 2 first paid at $t_1=0.085$. We show the skew just before, $T=0.33$ (solid), and after, $T=0.34$ (dashed), the second dividend. Exercise-style is American and $N=65$.

Volatility Curve Parametrization Wish List

- Parameters should have simple, intuitive meaning, esp. first three.
- Parameters should be “independent”, stable from day to day (parsimonious).
- Little term-structure, if possible.
- No-arbitrage constraints should be “easy” to incorporate.
- Parametric vols should be easy/fast to compute.
- No hacks! (in wings, etc)
- Vol curves arising from standard “SLVJ”-type model should be fittable within a few bps (at worst).

Benefits of good volatility curve/surface parametrization

- Can (pretty) easily be set/checked by humans if necessary/desired.
- With suitable fitting framework, have hope of producing fast and robust implied vol curves/surfaces.
- Portfolio level greeks via parameter bumps make sense for vanillas and exotics, and are east and fast to calculate.
- Local vols can be produced fast & robustly via Dupire formula.
- If we can fit all SLVJ models, can fit all real-world surfaces(?).

Our parametrization approach

- Work one term at a time, impose smoothness across terms.
- Factor out overall vol level (ATF) as: $\sigma_0 := \sigma(T, K = F)$.
- Define “shape” curve $f(z) = f(z|\mathbf{p})$ as function of **normalized strike (NS)**¹

$$z := \frac{y}{\hat{\sigma}_0} = \frac{\log(K/F)}{\sigma_0 \sqrt{T}}$$

such that

$$\sigma(z)^2 = \sigma_0^2 f(z|\mathbf{p})$$

- There are no standard definitions – we define dimensionless “skew” and “smile/convexity” as slope and curvature of shape curve:

$$f(z) =: 1 + s_2 z + \frac{1}{2} c_2 z^2 + \dots$$

¹For hybrid models with $D \neq 0$, use: $K/F \rightarrow (K - D)/(F - D)$.

Our parametrization approach (cont'd)

- s_2 and c_2 tend to have mild term-structure; they are even comparable across names. Have been range-bound for decades.
- Sometimes it is useful to work with s_1 , c_1 defined via

$$\sigma(z) =: \sigma_0 (1 + s_1 z + \frac{1}{2} c_1 z^2 + \dots)$$

- Trivially: $s_2 = 2s_1$, $c_2 = 2(c_1 + s_1^2)$.
- Note that

$$\sigma(z) = \sigma_0 + \frac{s_1}{\sqrt{T}} \log(K/F) + \dots,$$

so that an alternative definition of skew

$$\tilde{s}_1 := K \frac{\partial \sigma}{\partial K} \Big|_{K=F} = \frac{s_1}{\sqrt{T}}$$

- No simple relationships between alternative definitions of curvature/convexity/smile.

No-Arbitrage Constraints in Vol-Space

- No butterfly arbitrage: Implied density ρ should be positive

$$\hat{C}(T, K) = \int_0^{\infty} dS_T (S_T - K)_+ \rho_T(S_0 \rightarrow S_T)$$

$$\Rightarrow \partial_K^2 \hat{C}(T, K) = \rho_T(S_0 \rightarrow S)|_{S=K}$$

- No calendar arbitrage: Total BS variance $w(y) := T\sigma(y)^2$ has to be increasing in T at any fixed y .
- Necessary (but generally not sufficient) constraint on the asymptotic wing behavior of implied vols (R. Lee, 2004):

$$w(y) \leq 2|y| \quad \text{as } |y| \rightarrow \infty$$

Simple consequences: Implied density 1

- Local vols and implied densities can be calculated most neatly in terms of the total variance $w(y) = T\sigma(z)^2$. Eg the implied density:

$$\rho(y) = \frac{g(y)}{\hat{\sigma}(y)} n(d_-(y)),$$

where $n(x) = N'(x)$ is the normal density, and

$$g(y) = \left(1 - \frac{y w'(y)}{2w(y)}\right)^2 - \frac{1}{4} \left(\frac{1}{w(y)} + \frac{1}{4}\right) w'(y)^2 + \frac{1}{2} w''(y)$$

- Absence of butterfly arbitrage: $g(y) \geq 0$ for all y .
- In Black-Scholes case: $g(y) = 1$ for all y .
- NOTE: Will use same symbol whether we consider a quantity a function of z or y .

Simple consequences: Implied density 2

- For our vol curve parametrizations $w(y) = \hat{\sigma}_0^2 f(z)$.
- Then $w'(y) = \hat{\sigma}_0 f'(z)$ and $w''(y) = f''(z)$, so that

$$g(y) = \left(1 - \frac{z f'(z)}{2f(z)}\right)^2 - \frac{1}{4} \frac{f'(z)^2}{f(z)} - \frac{\hat{\sigma}_0^2}{16} f'(z)^2 + \frac{1}{2} f''(z) =: g(z).$$

- The vol level appears in only one place! All else only depends on shape parameters.
- Makes the analysis of butterfly arbitrage significantly simpler (but is still very hard in general).
- Will see example later for S3/SSVI curve.

ATF No-Arbitrage Constraints

- If $w(z) = \hat{\sigma}_0^2 (1 + s_2 z + \frac{1}{2} c_2 z^2 + \dots)$, then

$$g(z=0) = 1 + \frac{1}{2} c_2 - \frac{1}{4} s_2^2 (1 + \frac{1}{4} \hat{\sigma}_0^2)$$

- $g(0) \geq 0$ implies upper bound on slope

$$s_2^2 \leq \frac{4 + 2c_2}{1 + \frac{1}{4}\hat{\sigma}_0^2}$$

or lower bound on curvature ($c_1 = \frac{1}{2}c_2 - \frac{1}{4}s_2^2$)

$$c_1 \geq -1 + \frac{1}{16} s_2^2 \hat{\sigma}_0^2 \approx -1$$

- Very relevant around FOMC and earnings where not just $c_1 < 0$ but even $c_2 < 0$ can happen!

Specific Curves: Parabolas

- What are simplest possible curves? Need at least 3 parameters for ATF behavior.
- Vendors often use

$$\sigma(y)^n = \sigma_0^2 + s y + \frac{1}{2}c y^2 \quad (\text{or in terms of } z)$$

- Obviously has arbitrage in wings for $n = 1, 2$.
- Slight hope for $n = 4$, but would imply symmetric wings, which is intuitively and empirically wrong.
- Positivity has to be enforced too.
- Must do better...

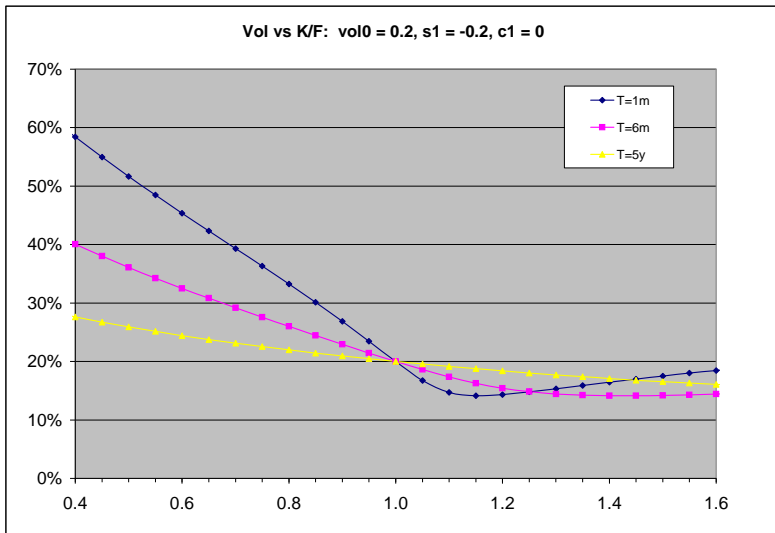
Specific Curves: S3/SSVI

- Simplest sensible curve with 3 parameters ($c_2 \geq 0$):

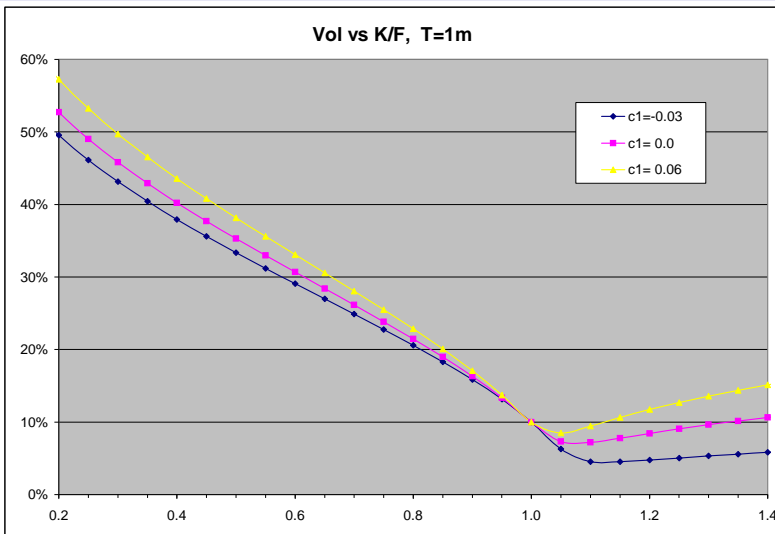
$$\sigma^2(z) = \sigma_0^2 \left(\frac{1}{2}(1 + s_2 z) + \sqrt{\frac{1}{4}(1 + s_2 z)^2 + \frac{1}{2} c_2 z^2} \right)$$

- Was independently discovered by TRK (2003, “S3”) and Gatheral/Jacquier (2013, “SSVI” = Simple SVI).
- Allows surprisingly varied skew shapes, including “takeover-for-cash” curves as $c_2 \rightarrow 0$. See **plots**.
- Allows fitting of vast majority of US equity names.
- Relatively easy to avoid (butterfly) arbitrage.
- In fact, in terms of the dimensionless variables $\hat{\sigma}_0, s_2, c_2$ can completely answer the butterfly-arbitrage question...

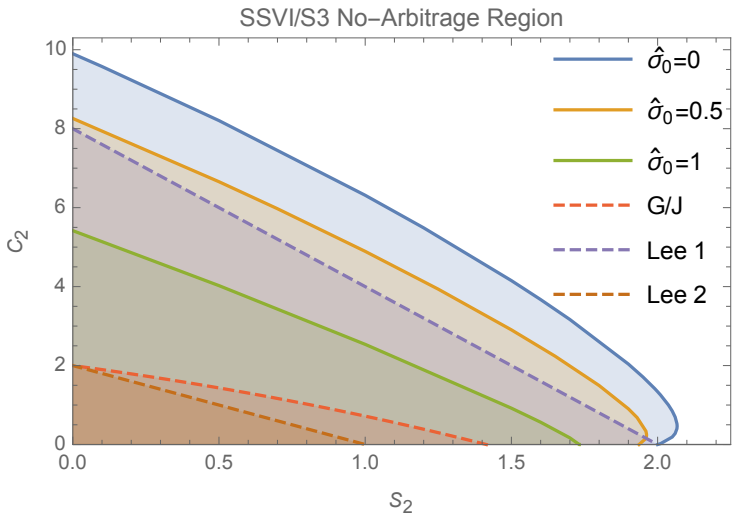
S3 shapes: different terms



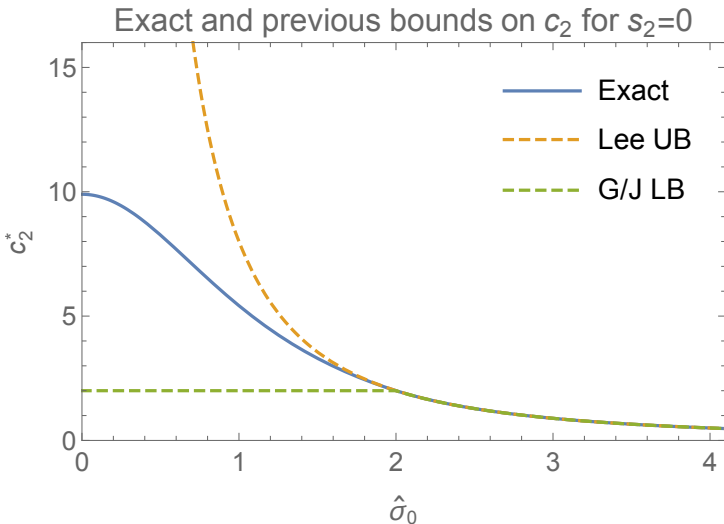
S3 shapes: different curvatures



Necessary and Sufficient No-Arb Conditions for S3/SSVI



Necessary and Sufficient No-Arb Conditions for S3/SSVI



Specific Curves: What to do for most liquid names?

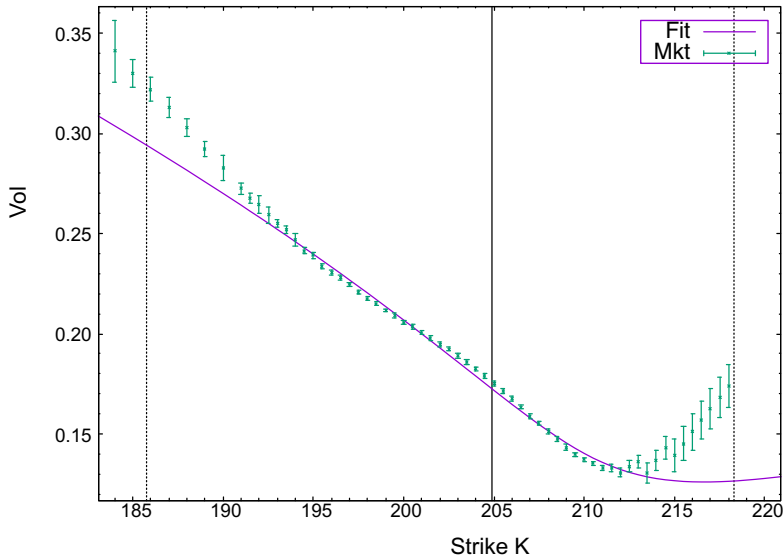
- For very liquid names (SPY, other ETFs, AAPL etc, KOSPI) none of the analytic curves (SVI, L5 or amendments) work well, even in the absence of events.
- There is a fundamental problem with the shapes allowed by these curves: Curvature has unique maximum around ATF, but that's not what the market wants! (Why?)
- Need more flexible shapes that can handle more generic curvature structures, incl. negative curvature around ATF: C5, C6, C7, C8.

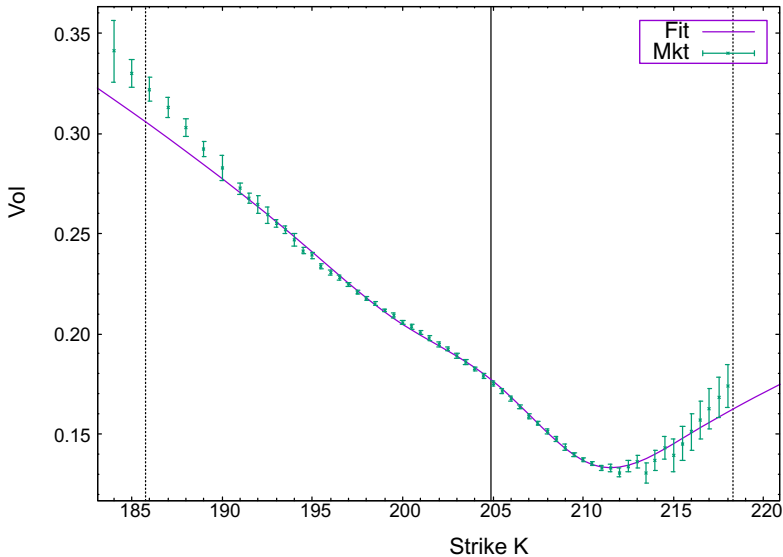
Volatility fitting framework

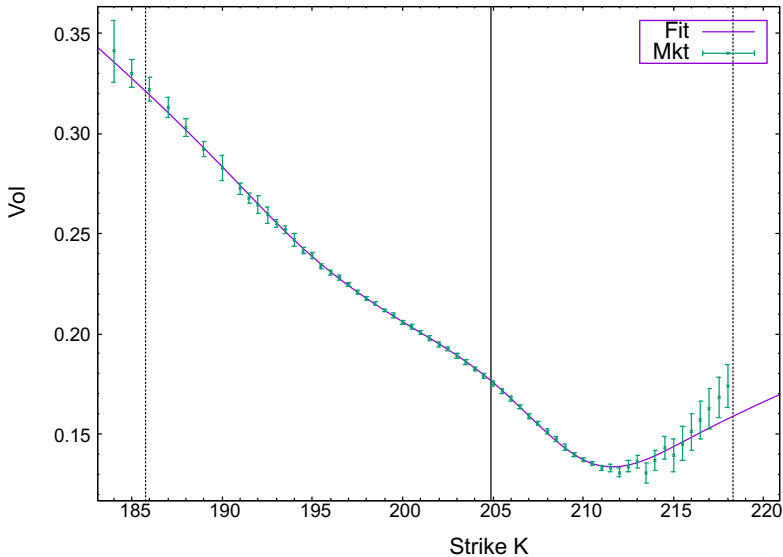
- Input to fitter are implied vols with error bars (after proper div modeling, borrow implication, etc).
- All our vol curves have sensible dimensionless parameters (first three are universal), which allows the use of curve-independent heuristics from 16 years of vol fitting experience across many names, geographies and asset classes.
- Fit one term at a time, transfer information between terms, for smoothness and stability.
- Minimize chi-square + soft penalties, for robustness and to allow the fitting of terms with less (effective) data than parameters.
- Good microprices help, but even then various heuristics are needed to deal with data issues in real-time.
- Keeping track of quality-of-fit metrics and error bars for final outputs is crucial for real-time trading applications.

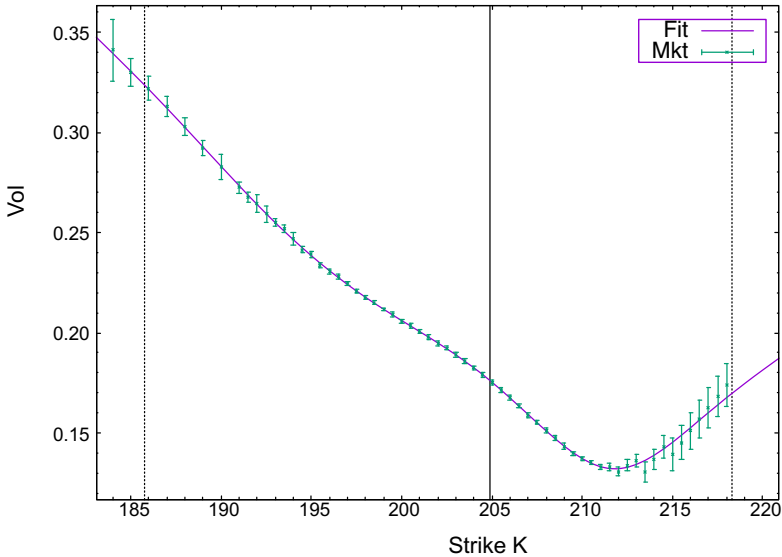
Volatility fitting examples

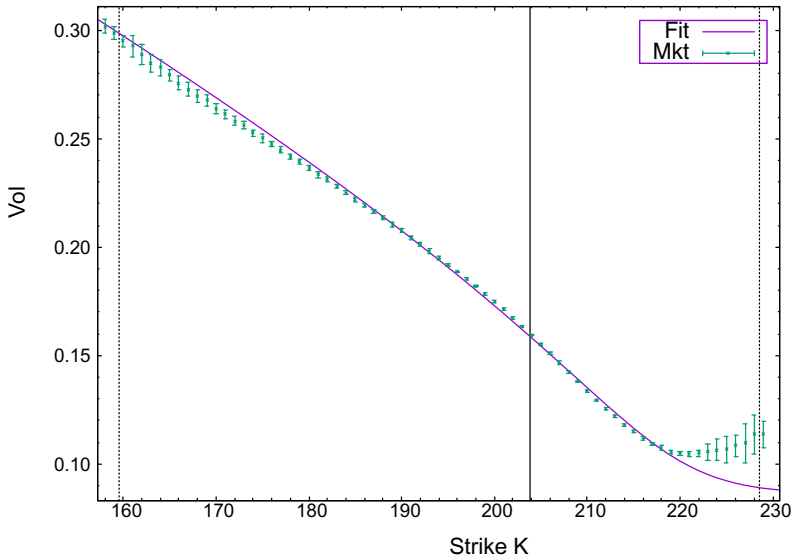
- Examples are fits of American-style options on liquid US ETFs or stocks (plus E-mini futures options).
- Starting with options and underlier prices, we need to:
 - pick interest rate
 - pick cash divs (if appropriate)
 - imply borrow cost for each term to get “American PCP”
 - imply vol-by-strike
 - fit all terms to various vol curves
- Are using simple mid for prices; vol error bars come from bid-ask spread in price space.
- Equity option price data were provided by MayStreet LLC.

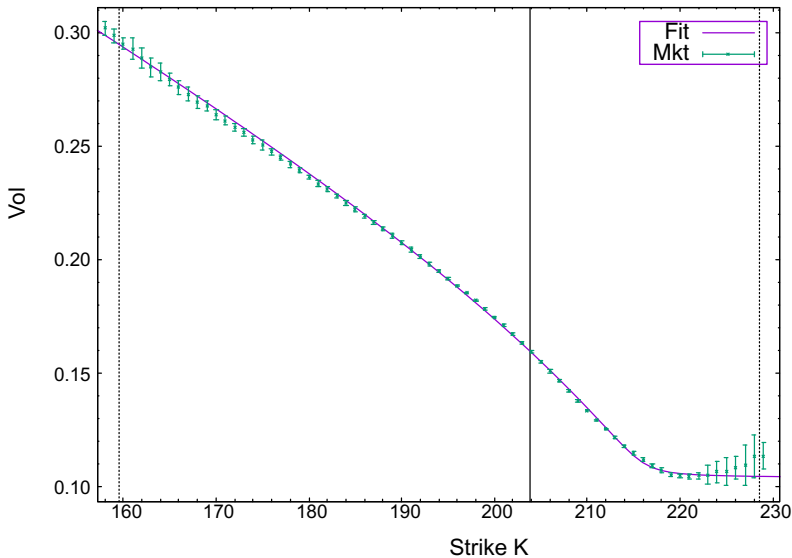
SPY 20150820-154500 S3: $T=0.0221$, $i=1$, $\text{chi}=6.931$, $\text{avE5}=34$ 

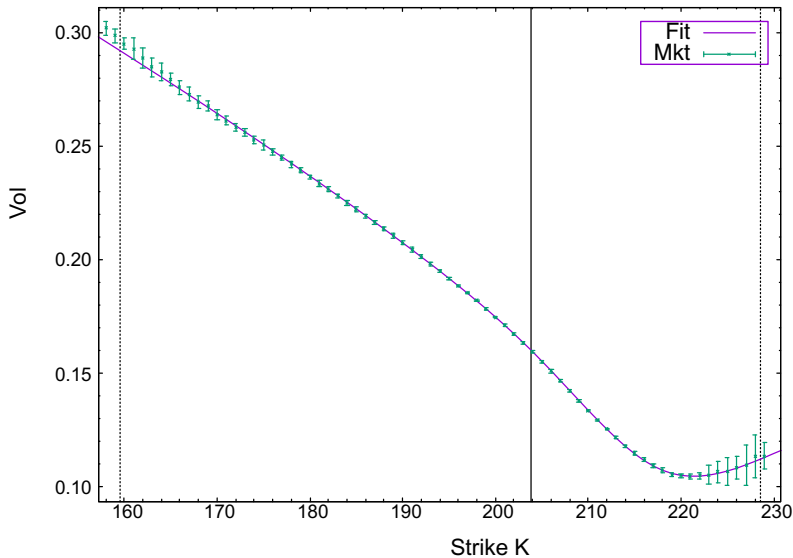
SPY 20150820-154500 C5: $T=0.0221$, $i=1$, $\chi=1.344$, $\text{avE5}=7$ 

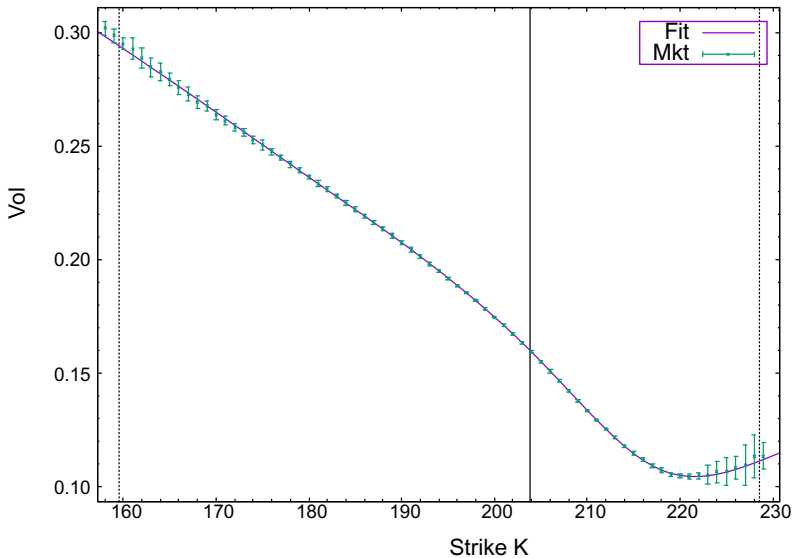
SPY 20150820-154500 C6: $T=0.0221$, $i=1$, $\chi=0.306$, $avE5=4$ 

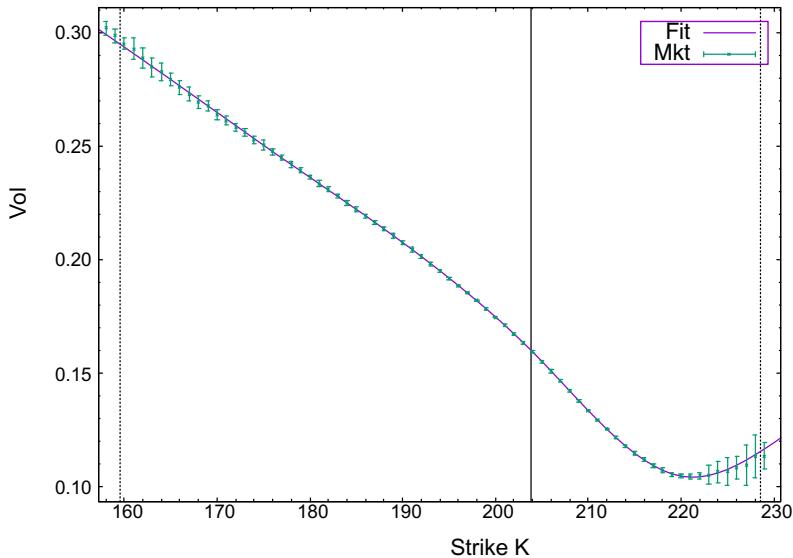
SPY 20150820-154500 C8: $T=0.0221$, $i=1$, $\chi=0.153$, $avE5=1$ 

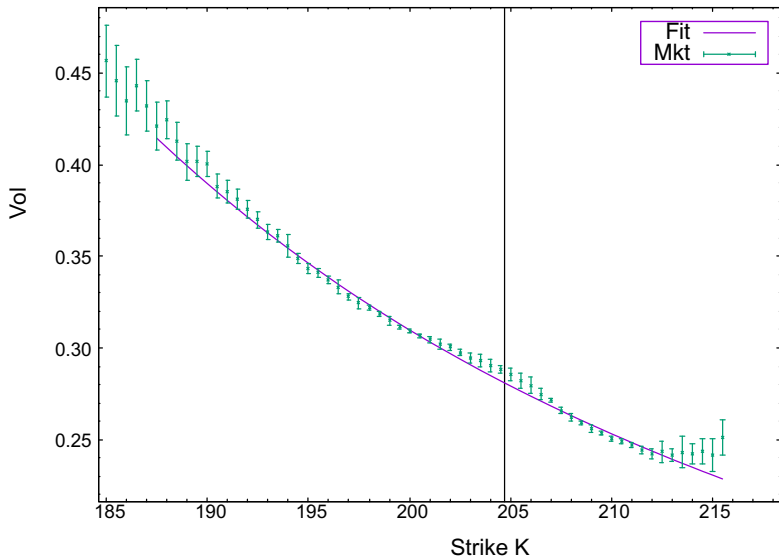
SPY 20150820-154500 S3: $T=0.1564$, $i=8$, $\chi=6.705$, $avE5=14$ 

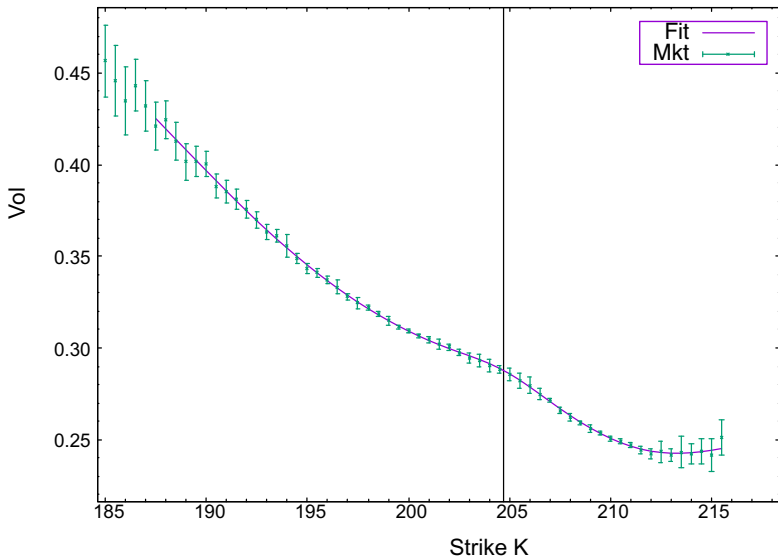
SPY 20150820-154500 SVI5: $T=0.1564$, $i=8$, $\chi=1.541$, $avE5=5$ 

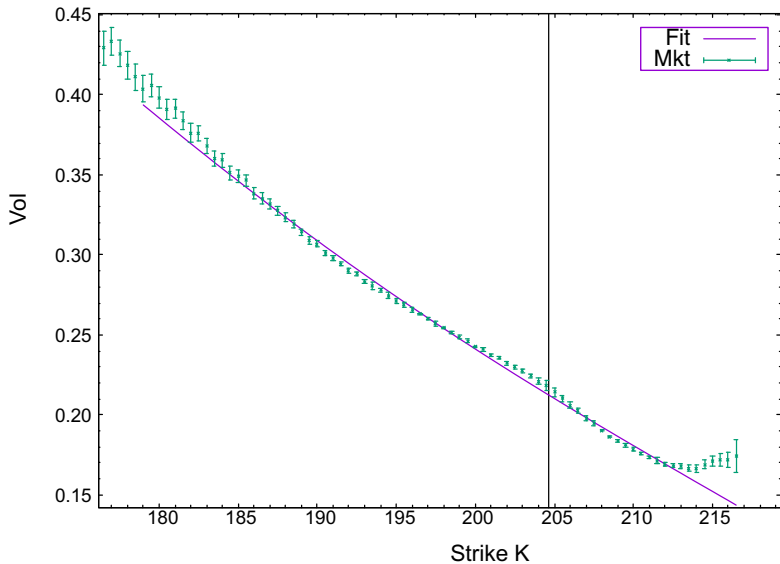
SPY 20150820-154500 C5: $T=0.1564$, $i=8$, $\chi=0.147$, $avE5=1$ 

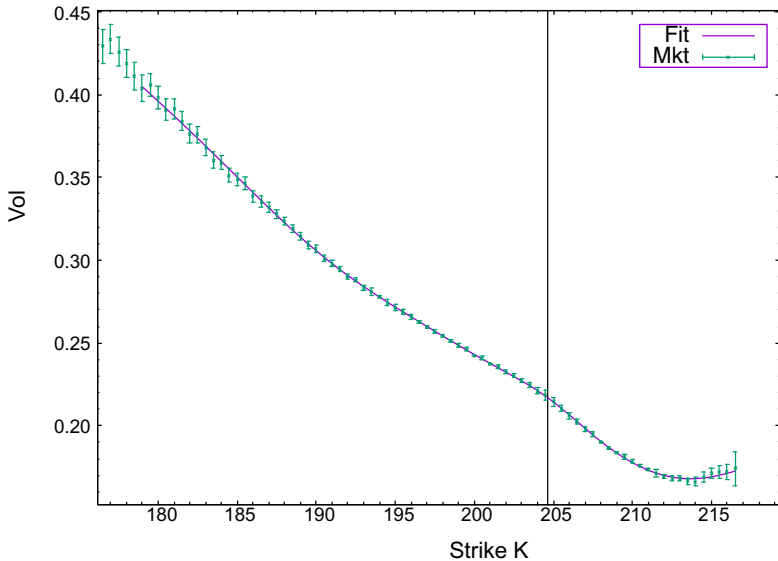
SPY 20150820-154500 C6: $T=0.1564$, $i=8$, $\chi=0.077$, $avE5=0$ 

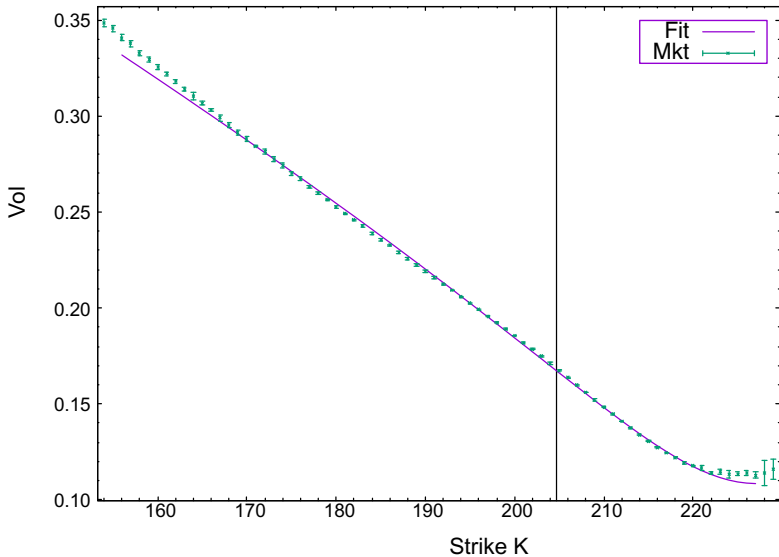
SPY 20150820-154500 C8: $T=0.1564$, $i=8$, $\chi=0.052$, $avE5=0$ 

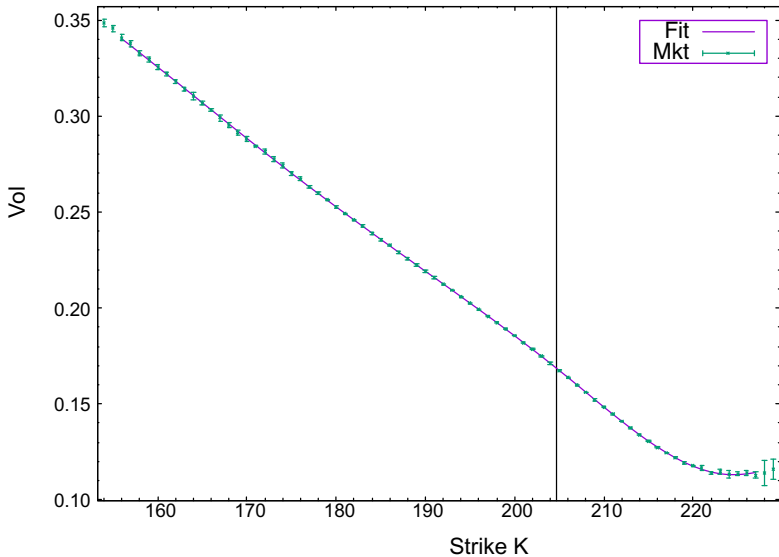
SPY 20151216-124500 SVI5: $T=0.0088$, $i=0$, $\chi=1.879$, $avE5=60$ 

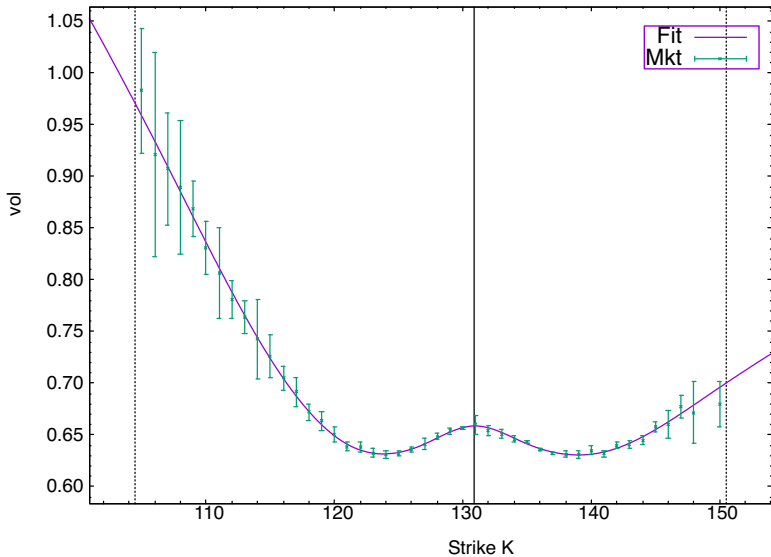
SPY 20151216-124500 C8: $T=0.0088$, $i=0$, $\chi=0.143$, $avE5=6$ 

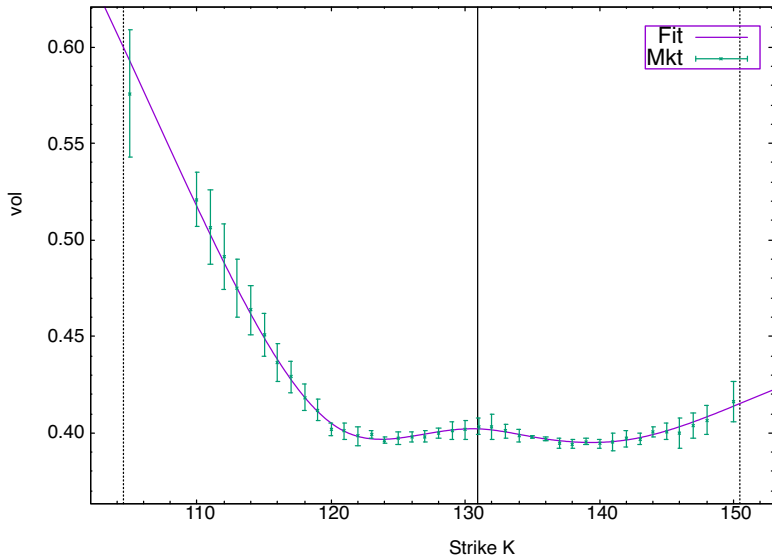
SPY 20151216-124500 SVI5: $T=0.0225$, $i=1$, $\chi=6.961$, $avE5=37$ 

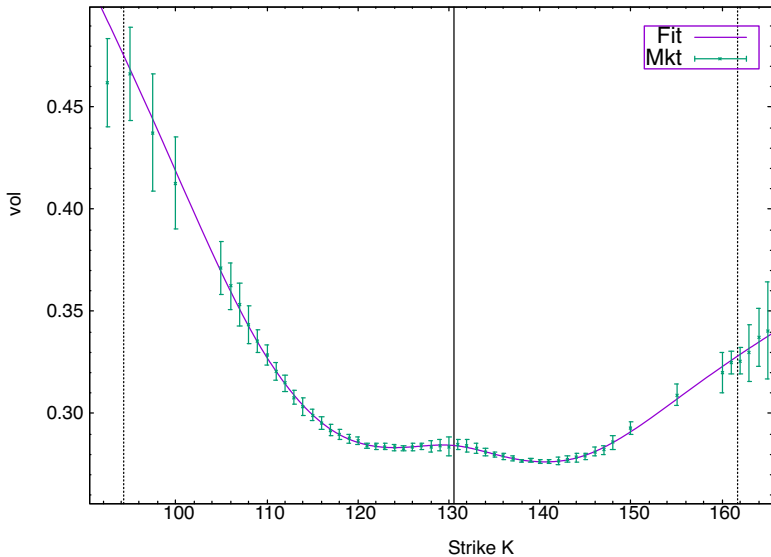
SPY 20151216-124500 C8: $T=0.0225$, $i=1$, $\chi=0.213$, $avE5=2$ 

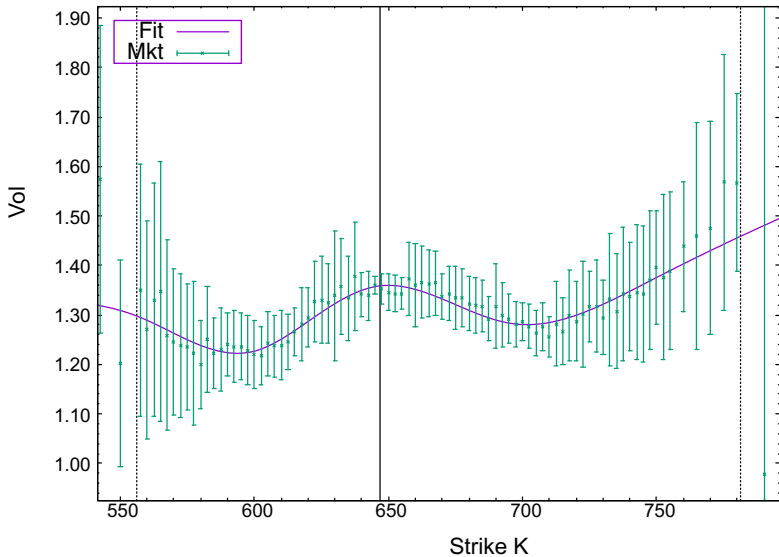
SPY 20151216-124500 SVI5: $T=0.1786$, $i=7$, $\chi=9.325$, $avE5=13$ 

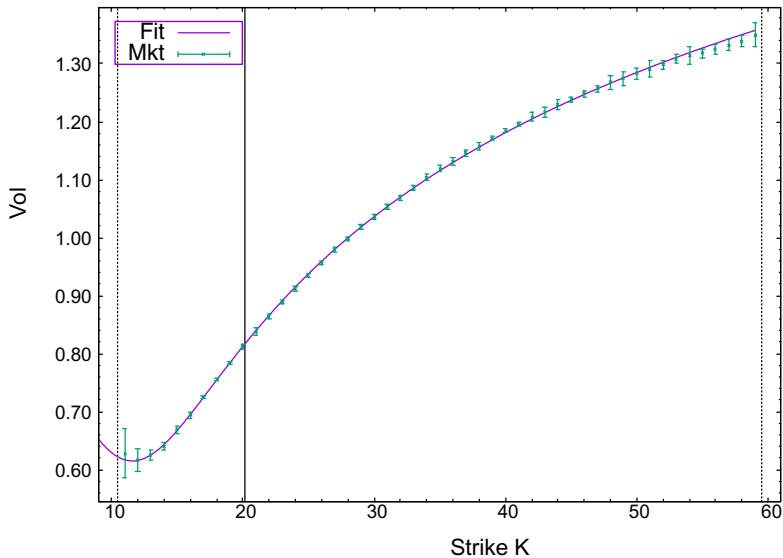
SPY 20151216-124500 C8: $T=0.1786$, $i=7$, $\chi=0.124$, $avE5=0$ 

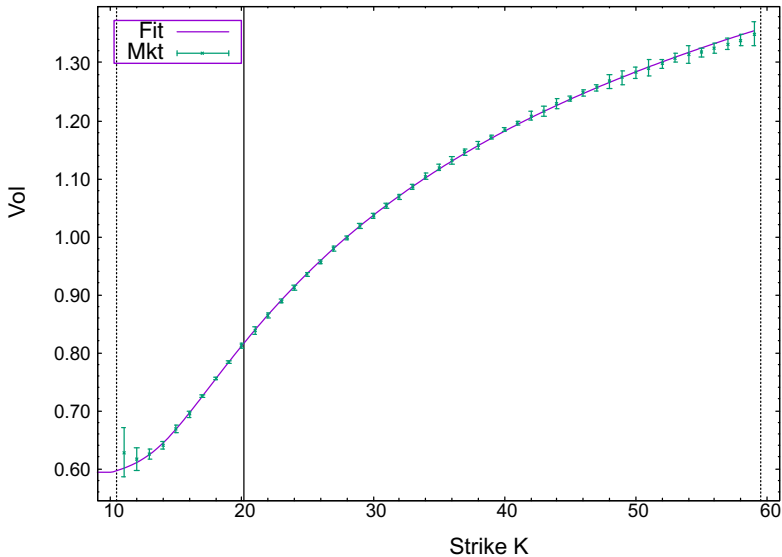
AAPL 20150721-154500 C8: $T=0.0084$, $i=0$, $\chi=0.247$, $avE5=13.2$ 

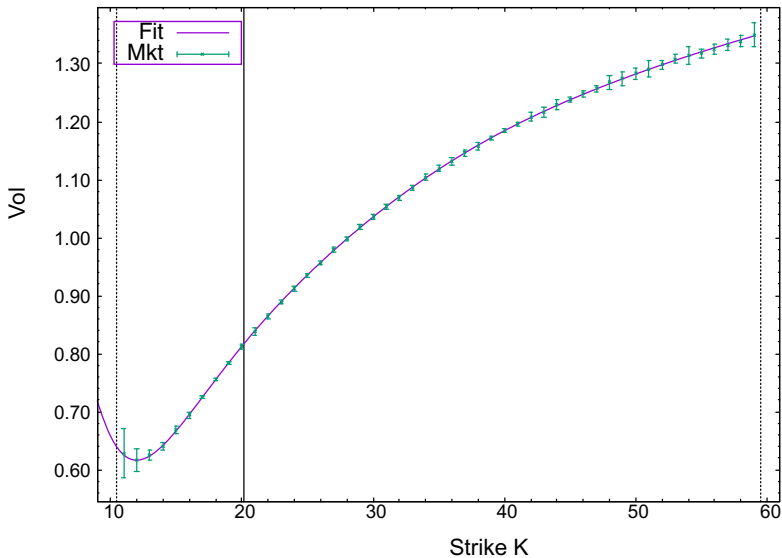
AAPL 20150721-154500 C8: $T=0.0276$, $i=1$, $\chi=0.150$, $avE5=8.0$ 

AAPL 20150721-154500 C8: $T=0.0851$, $i=4$, $\chi=0.060$, $avE5=5.4$ 

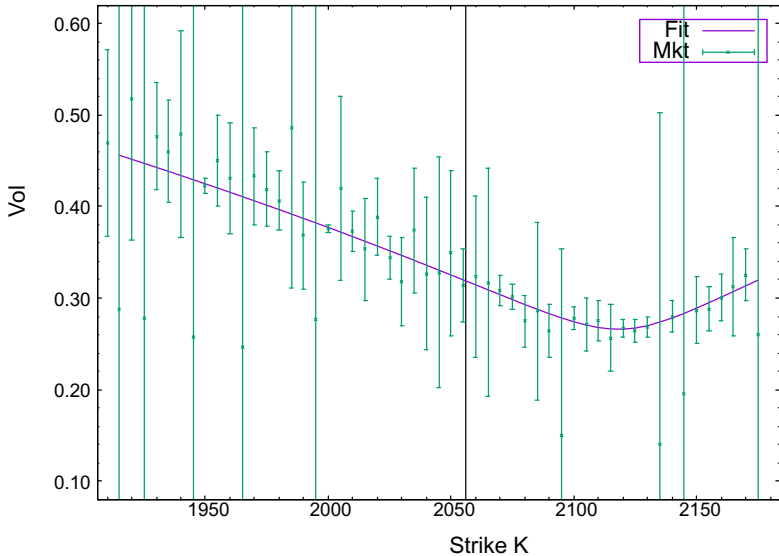
GOOG 20151022-150000 C6: $T=0.0030$, $i=0$, $\chi=0.063$, $avE5=90$ 

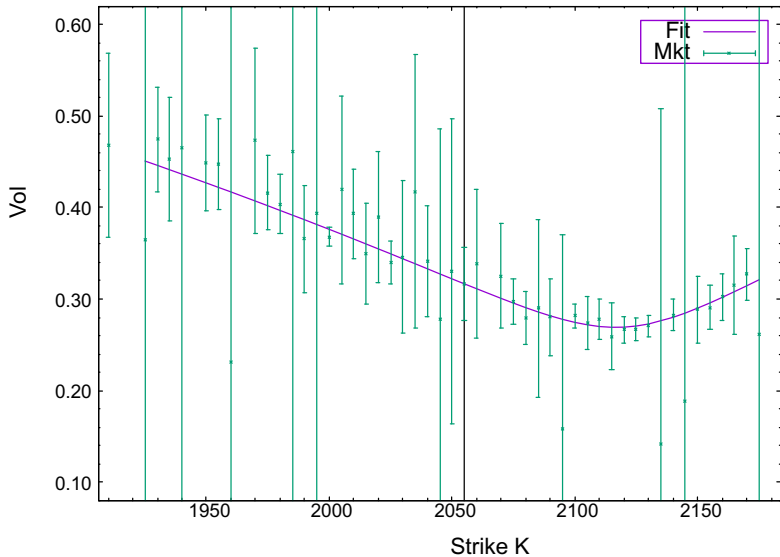
VXX 20151216-134500 SVI5: $T=0.1784$, $i=7$, $\chi=0.282$, $avE5=12$ 

VXX 20151216-134500 C5: $T=0.1784$, $i=7$, $\chi=0.205$, $avE5=10$ 

VXX 20151216-134500 C7: $T=0.1784$, $i=7$, $\chi=0.055$, $avE5=9$ 

ESZ5 20151216-135901 SVI5: $T=0.0050$, $i=0$, $\chi=0.137$, $avE5=99$



ESZ5 20151216-135951 SVI5: $T=0.0050$, $i=0$, $\chi=0.126$, $avE5=219$ 

Summary and Conclusion

- There is no standardization in the equity options markets around dividend modeling, borrow costs, or vol curves and their calibration.
- No borrow or vol curves are publicly available, historical or live, free or for purchase!
- No vol curves in the public domain can fit liquid names like SPY and AAPL. (Some believe they can only be fit non-parametrically...)
- Superior modeling and numerical expertise are still crucial for fast and robust real-time options valuation.
- Lack of transparency hinders the wider use of options and the efficient transfer of vol information across related products.
- Equity options are due for some major “RND”:
Rationalization, Normalization and Democratization!
- Hopefully can achieve same as in transition from old to new VIX:
A healthier market, larger volumes, esp. from smaller players.
- Want to help? Stay tuned!

VOLAR – What we do

- Super-fast, robust, and sensible pricing, fitting, and volatility curve analytics. To start:
 - Drop-in replacements for pain points in most firms' infrastructure:
pricer and **fitter** (simple API, hard analytics underneath)
- Provide, in real-time and historical, all valuation, risk, and trade analysis data and services relevant for options trading firms.
- Consulting, custom design and development
- For more information: info@volar.io