Optimal portfolio construction and risk premia in options markets

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VOLA DYNAMICS
Analytics for Options Trading

Motivation

- Long equities, short vol, short skew positions behave similar during risk on/off scenarios.
- Are these strategies the same bet? Is one better than the others?
- Given many listed expiries, is selling vol/skew in one of them better than in the others?
- How should one think about it? How should one trade?

Outline

- Optimal options portfolio
- Optimal portfolio performance in comparison to forwards and straddles
- Efficient market in the presence of a profit opportunity
- Relationship between equity, volatility, and skew risk premia
- Term structure of volatility risk premium

Optimal options portfolio

- $\rho_Q(S)$ implied density under the risk-neutral measure Q (calculated from the implied vol curve)
- $\rho_P(S)$ forecasted density under the real measure P (forecast spot, vol, skew etc.)

How should you structure your portfolio?

Optimal options portfolio (continued)

Future market value of a contact with the payoff $f(S)$:

$$V = \int \rho_Q(S')f(S')dS'$$

Profit and loss in future dollars as a function of the stock price on expiration:

$$PnL(S) = f(S) - V$$

Maximize the expected rate of return or log of wealth (leads to more wealth almost surely in the long run):

$$U_f(S) = \log(NAV + PnL(S))$$

Optimal payoff

Expected utility under the real measure is

$$u_f = \langle U_f(S) \rangle_P = \int \rho_P(S) log(NAV + f(S) - V) dS$$

Maximization gives the following payoff:

$$\boxed{f(S) = NAV \left(\frac{\rho_P(S)}{\rho_Q(S)} - 1 \right)}$$

How to compare strategies/payoffs?

- Expected return (the correct way)

$$\langle \boldsymbol{\mu} \rangle = \int \boldsymbol{\rho_P(S)} \frac{\log\left(1 + \frac{PnL(S)}{NAV}\right)}{T} d\boldsymbol{S}$$

For payoffs like leveraged forward and short straddle $\langle \mu \rangle = -\infty$, so for illustration, we modify the definition by flooring the PnL:

$$\langle \mu \rangle = \int \rho_P(S) \frac{\log\left(max\left(0.01, 1 + \frac{PnL(S)}{NAV}\right)\right)}{T} dS$$

Also commonly used (slightly modified):

- Sharpe ratio

$$\langle \text{pnl} \rangle = \int \rho_P(S)\, PnL(S) dS$$

$$std^2 = \int \rho_P(S)\, (PnL(S) - \langle \text{pnl} \rangle)^2 dS$$

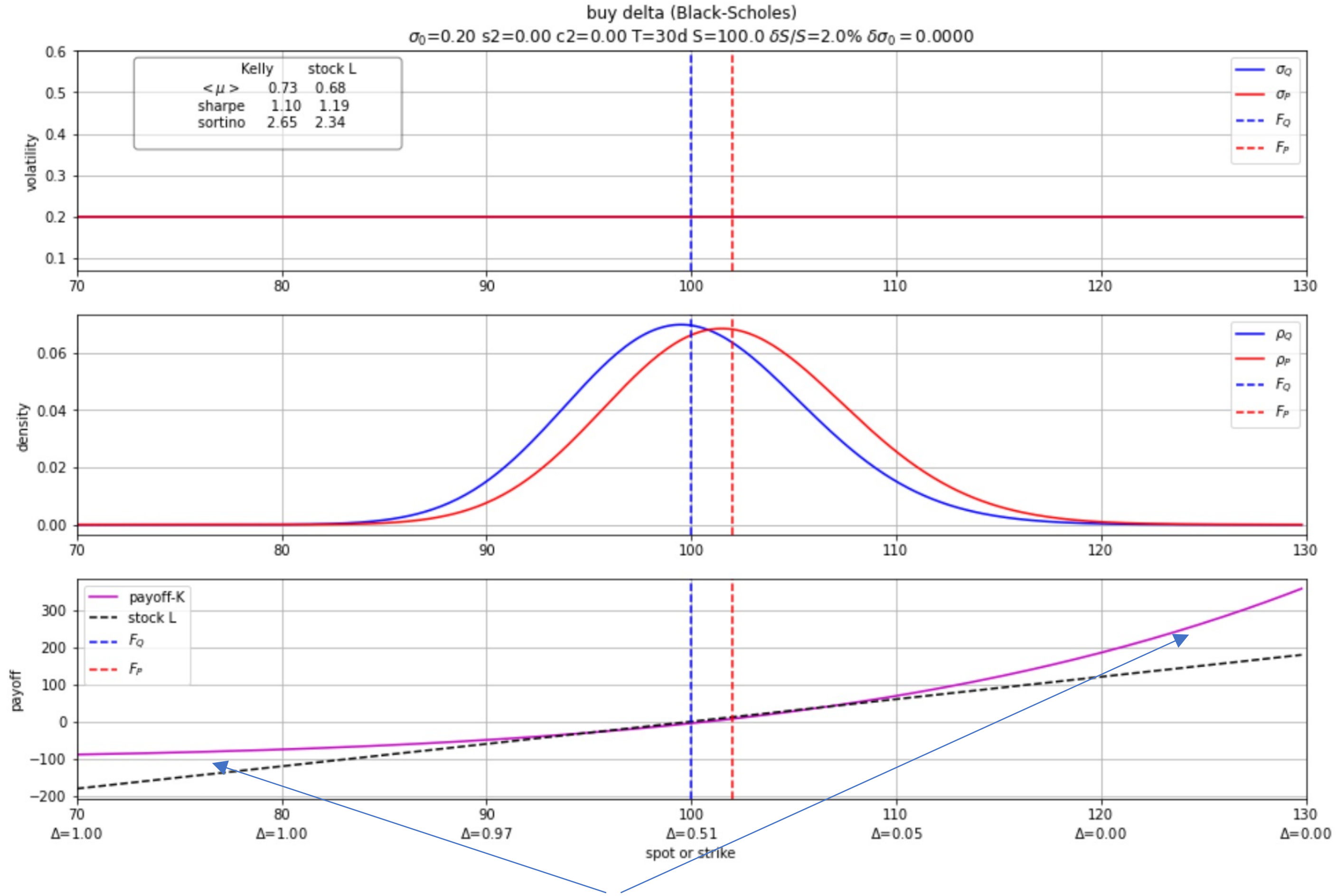
$$sharpe = \frac{\langle \text{pnl} \rangle}{std\sqrt{T}}$$

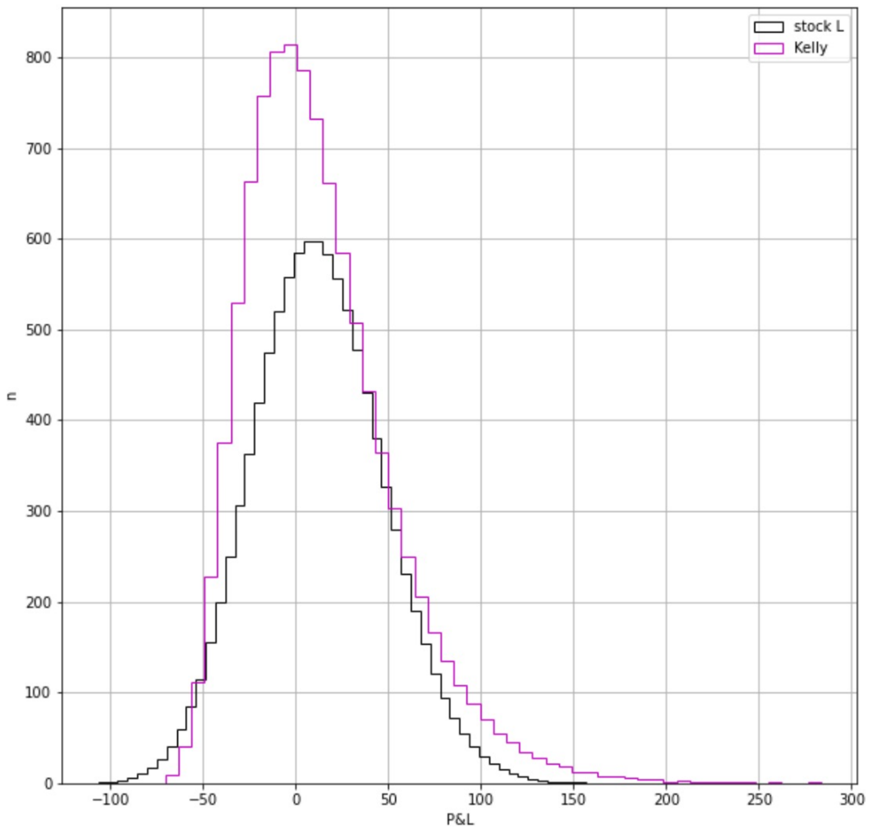
- Sortino ratio

$$std_d^2 = \int \rho_P(S)\, min(0, f(S) - V)^2 dS$$

sortino= $\frac{\langle \text{pnl} \rangle}{std_d\sqrt{T}}$

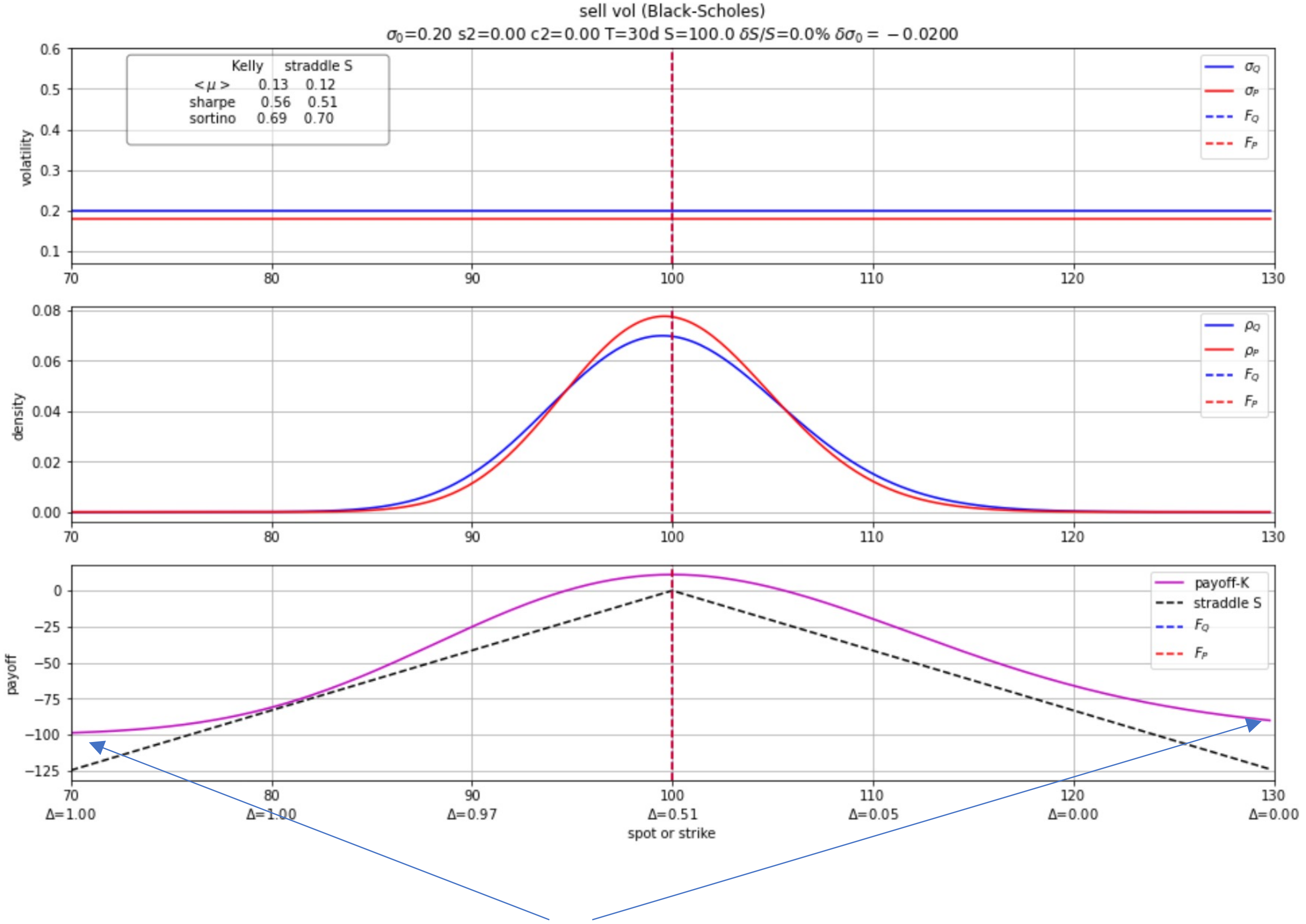
Black-Scholes, cheap spot, fairly(?) priced vol
Kelly payoff vs Forward with the same delta

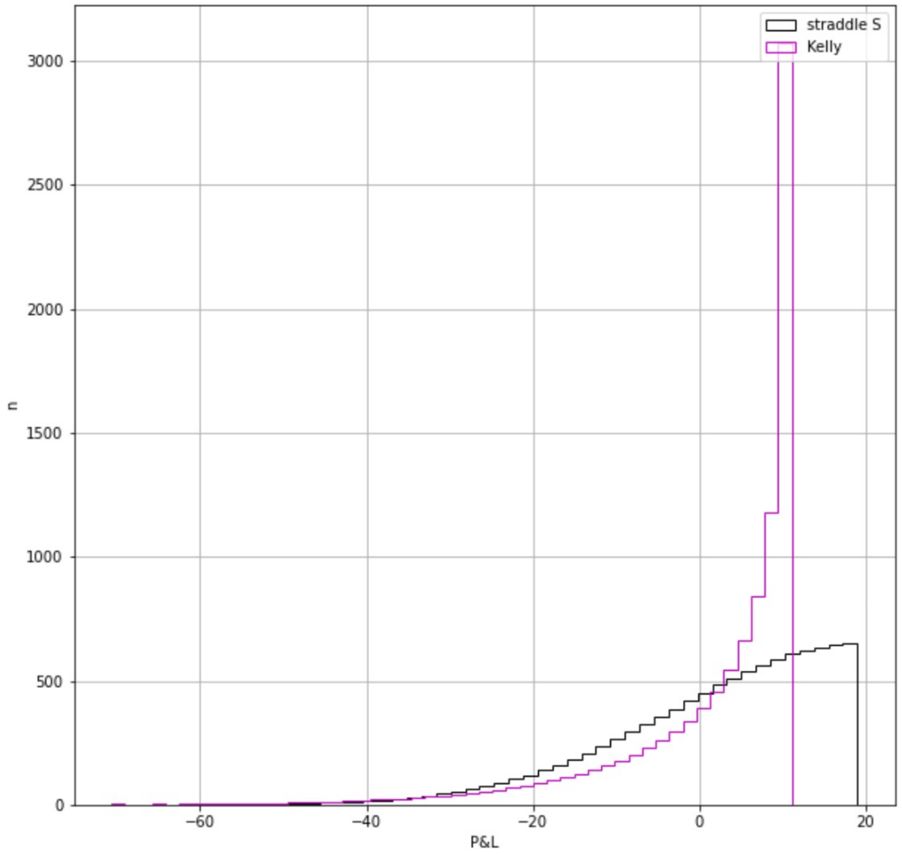




Buy options in the tails to improve (risk adjusted) returns!

Black-Scholes, rich vol
Kelly payoff vs straddle with the same vega





Buy options in the tails – never lose everything

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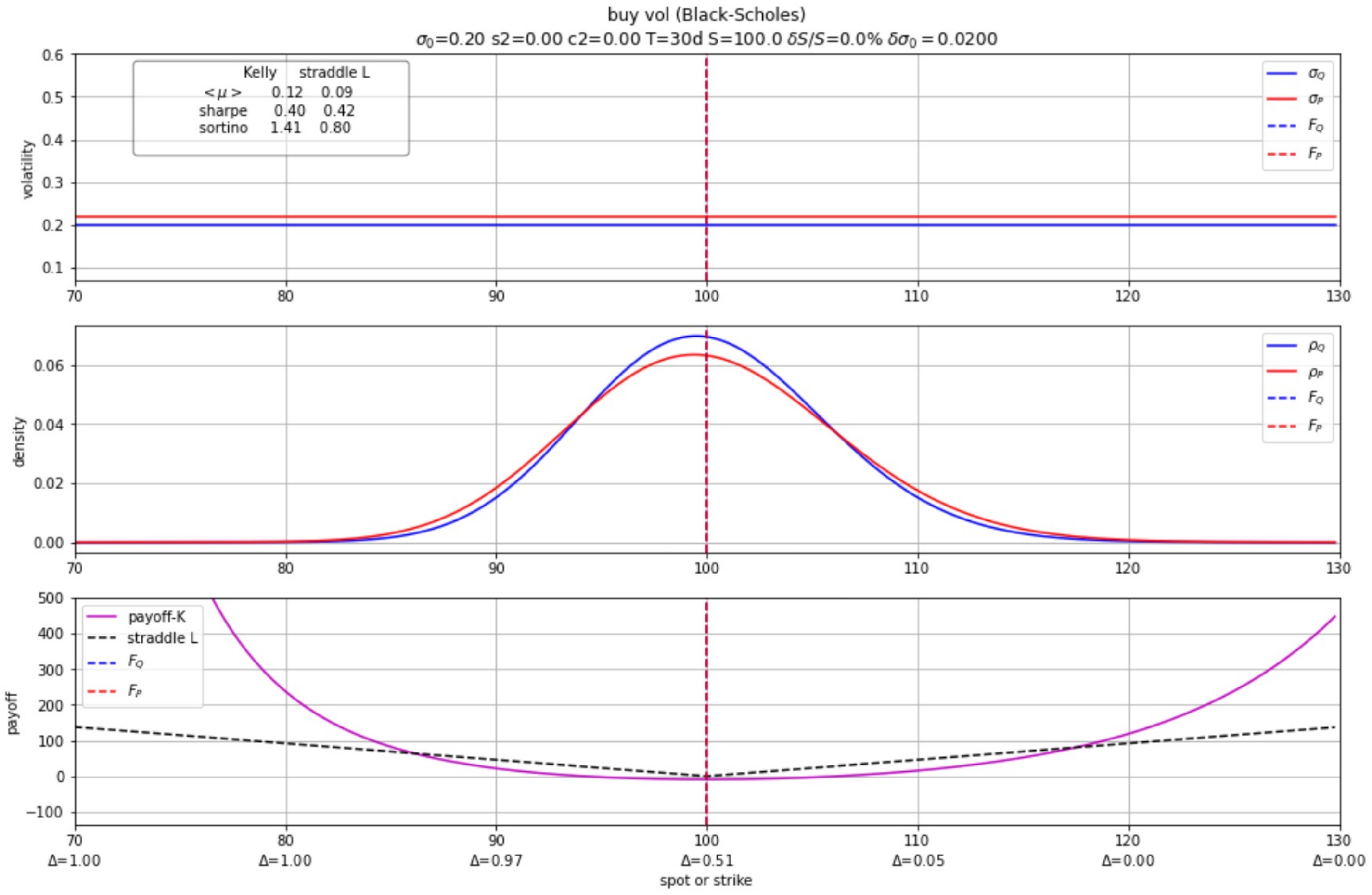
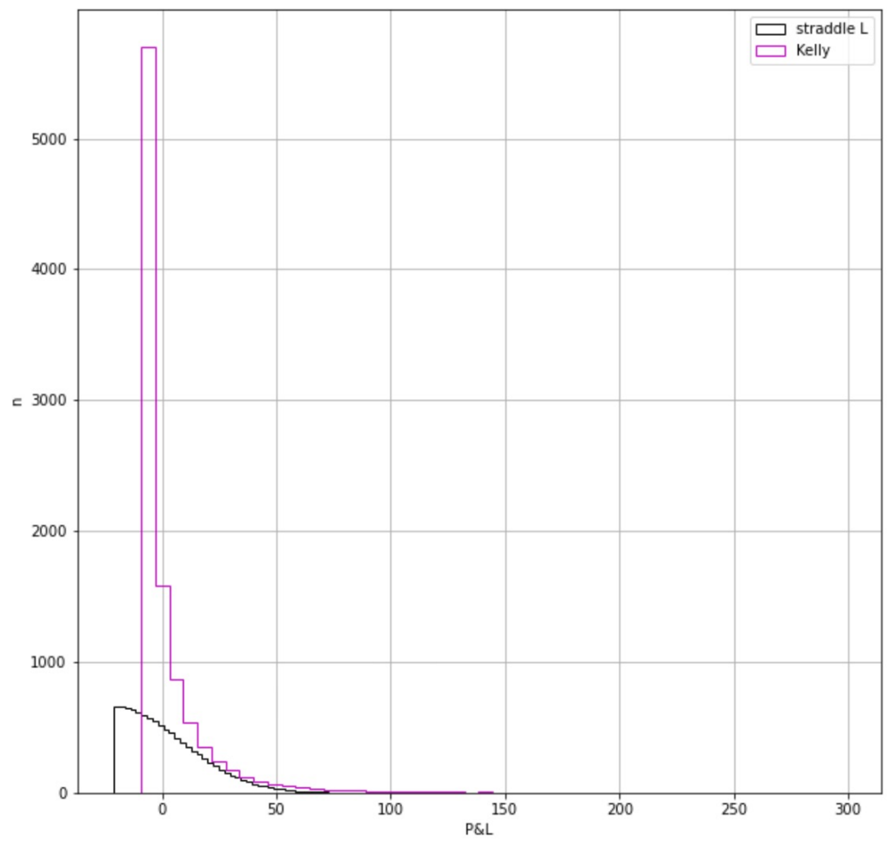
Black-Scholes, cheap vol
Kelly payoff vs straddle with the same vega



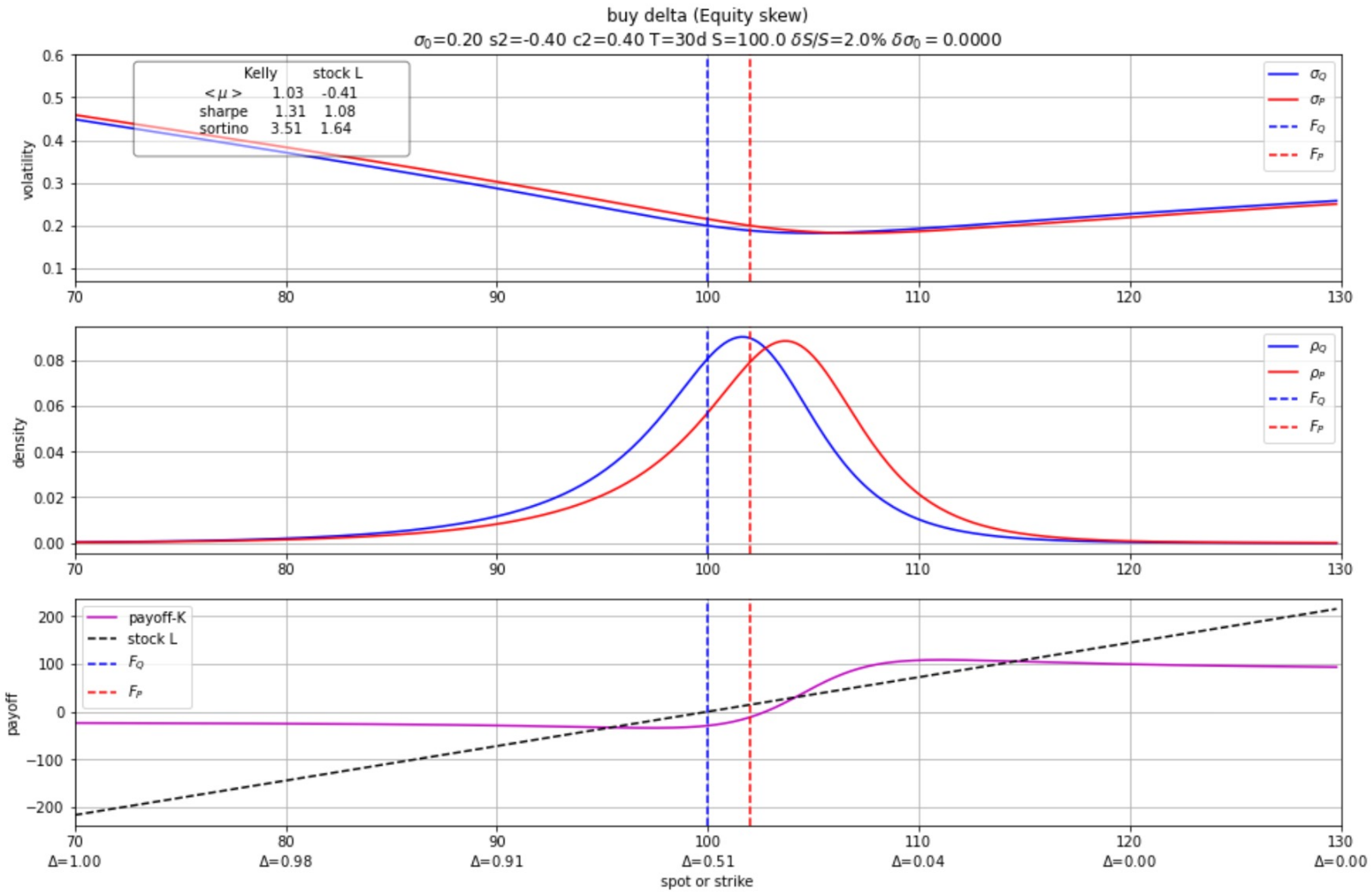


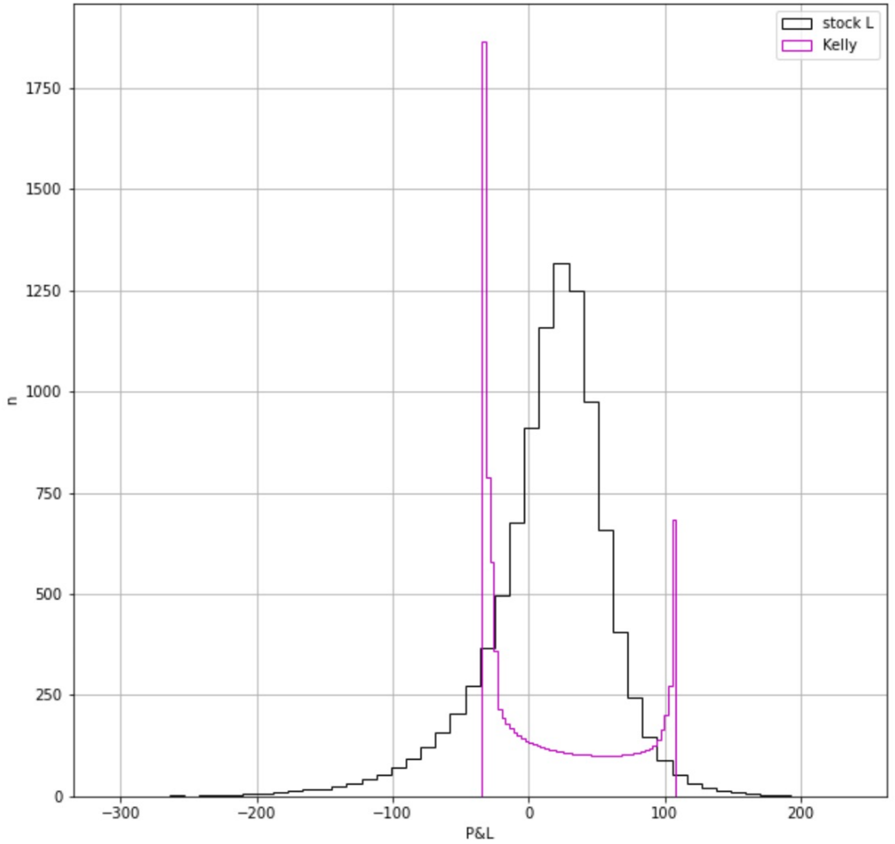
Optimal portfolios in Black-Scholes

Sure, Kelly-optimal portfolios never blow up, but the improvement in the average case is not impressive, right?

Let's add fat tails…

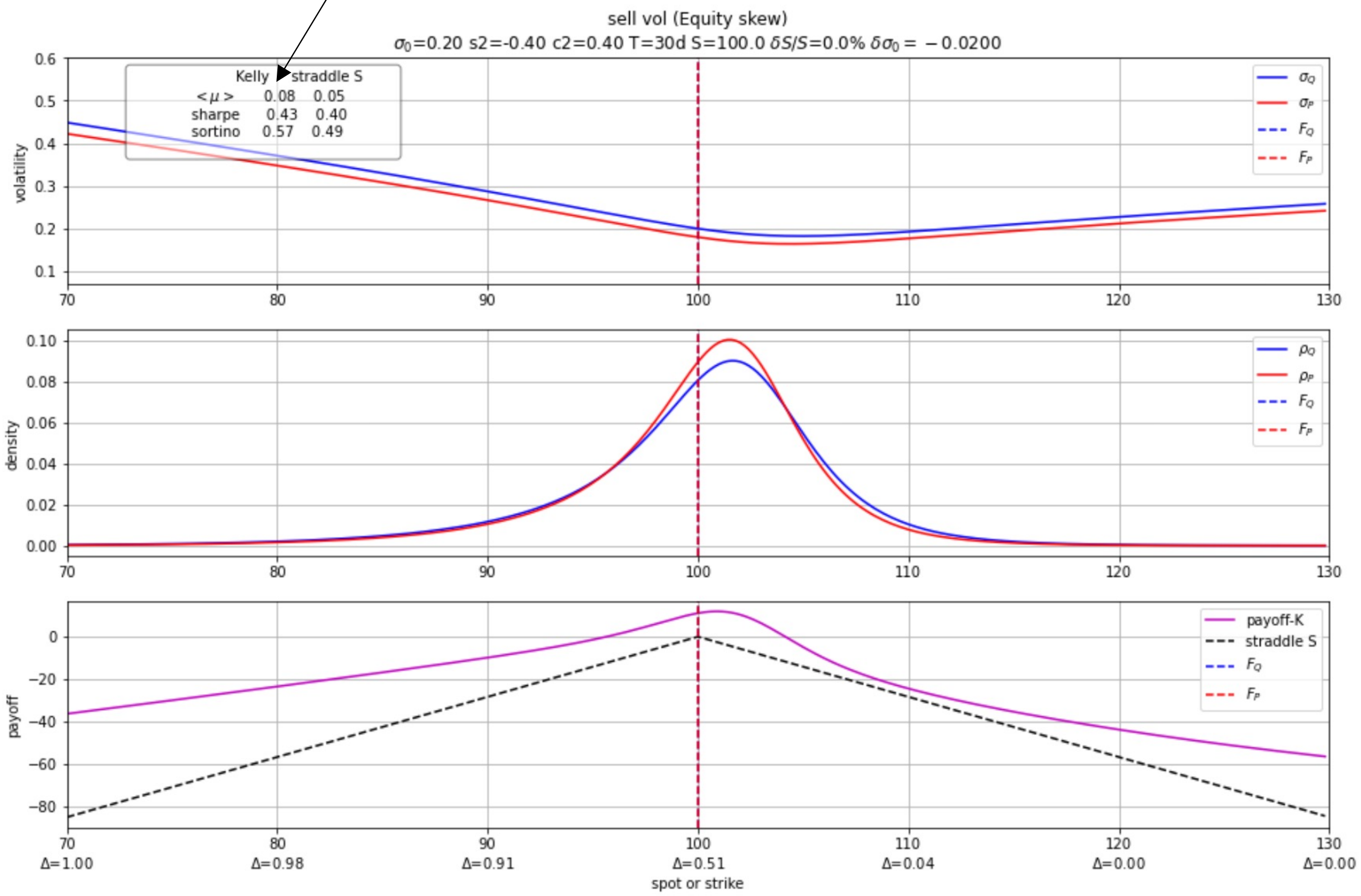
Equity skew, cheap spot, fairly(?) priced vol Kelly payoff vs Forward with the same delta

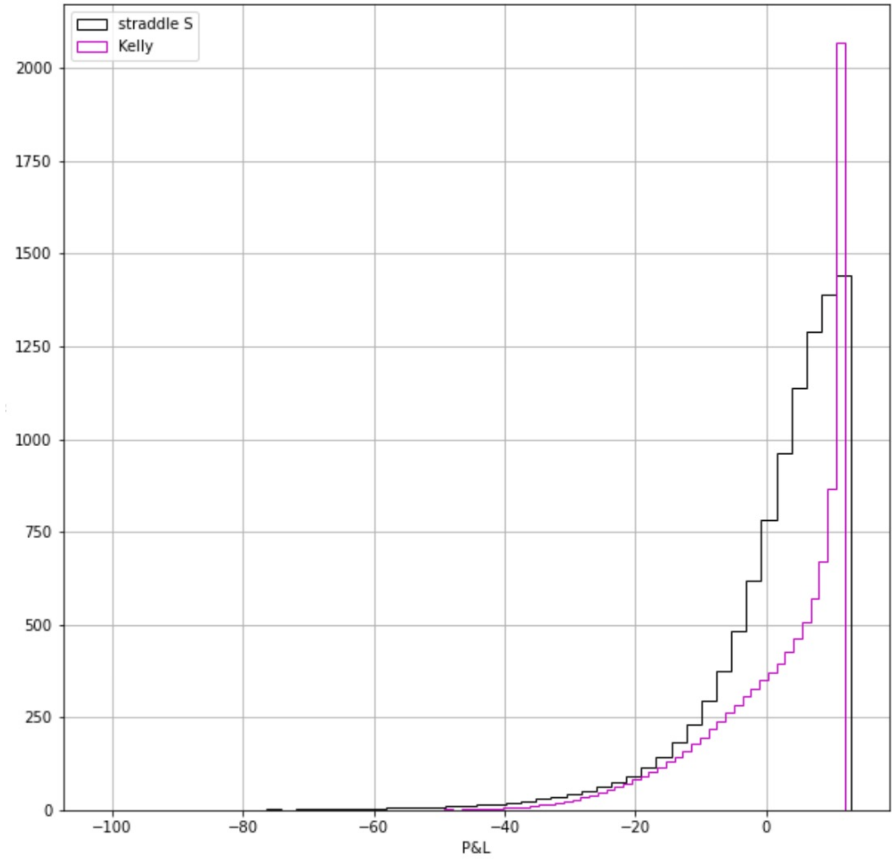




Equity skew, rich vol
Kelly payoff vs straddle with the same vega

Notice that is hard to achieve high returns selling vol even when doing full Kelly!





Implication for risk premium

Equity risk premium: expected return is greater than the risk-free rate:

$$\int \rho_P(S')S'dS' > \int \rho_Q(S')S'dS' = F = S_0 e^{r\,T}$$

If investing with a forward (without rebalancing), the optimal leverage *b* is given by:

$$b = \ argmax\, u(b) = \int \rho_P(S) log\left(1 + \left(\frac{S}{F} - 1\right) b\right) dS$$

To solve for *b*:

$$\int \rho_P(S) \frac{1}{1 + \left(\frac{S}{F} - 1\right) b} dS = 1$$

Options pricing and equity risk premium

How should one price options in the presence of equity risk premium?

No free lunch in the presence of profit opportunity:

> Equilibrium options valuations should be such that one cannot improve the average rate of return of a delta-one portfolio by trading options.

Note, this is similar to indifference pricing in incomplete markets.

If one could improve the performance by trading options, then trading in such options would push the market to the equilibrium as long as there are players maximizing expected returns.

This means that the optimal options payoff is the forward:

$$f(S) = NAV\left(\frac{\rho_P(S)}{\rho_Q(S)} - 1\right) = NAV\ b\left(\frac{S}{F} - 1\right)$$

Fair risk-neutral measure

Then the implied probability density is given by

$$\rho_Q(S) = \frac{\rho_P(S)}{1 + \left(\frac{S}{F} - 1\right) b}$$

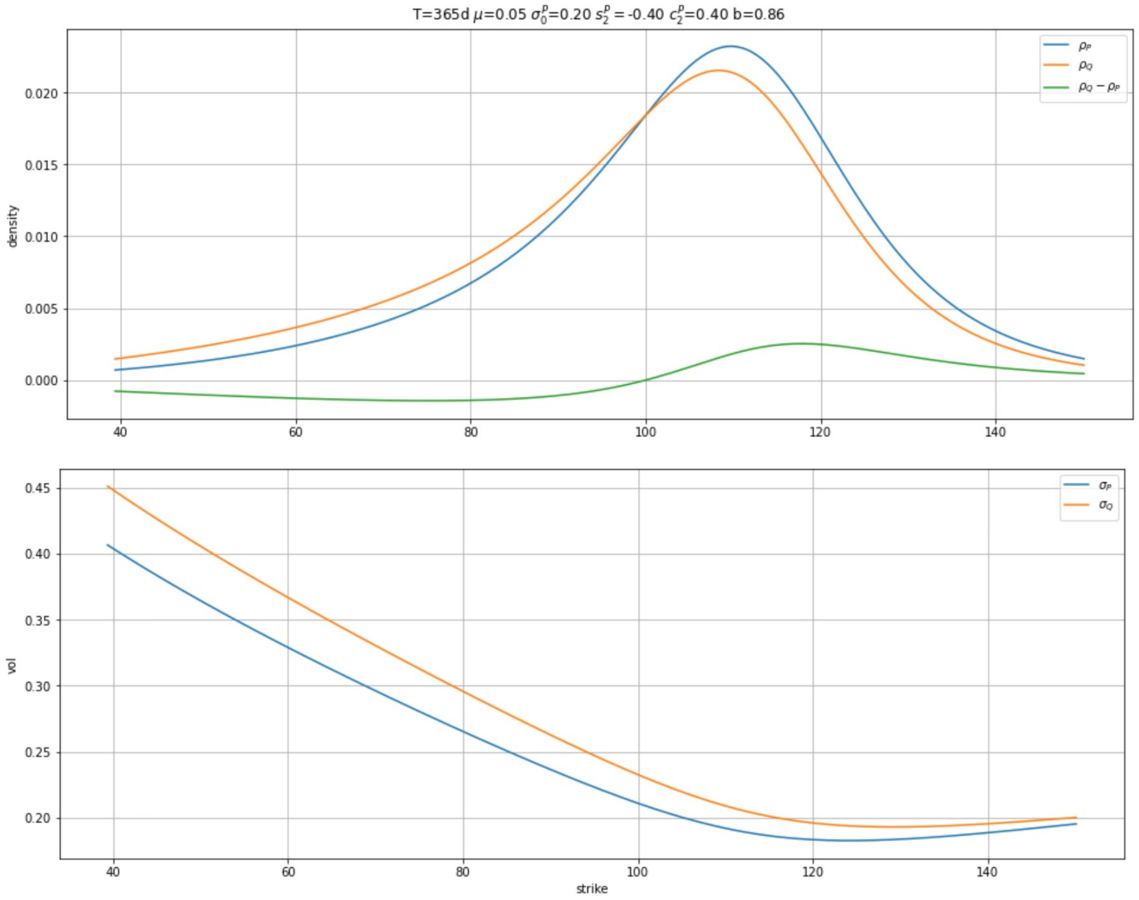
It is easy to verify that

$$\int \rho_Q(S)\, dS = 1$$

and

$$\int \rho_Q(S)\, S\, dS = F$$

Risk premium example

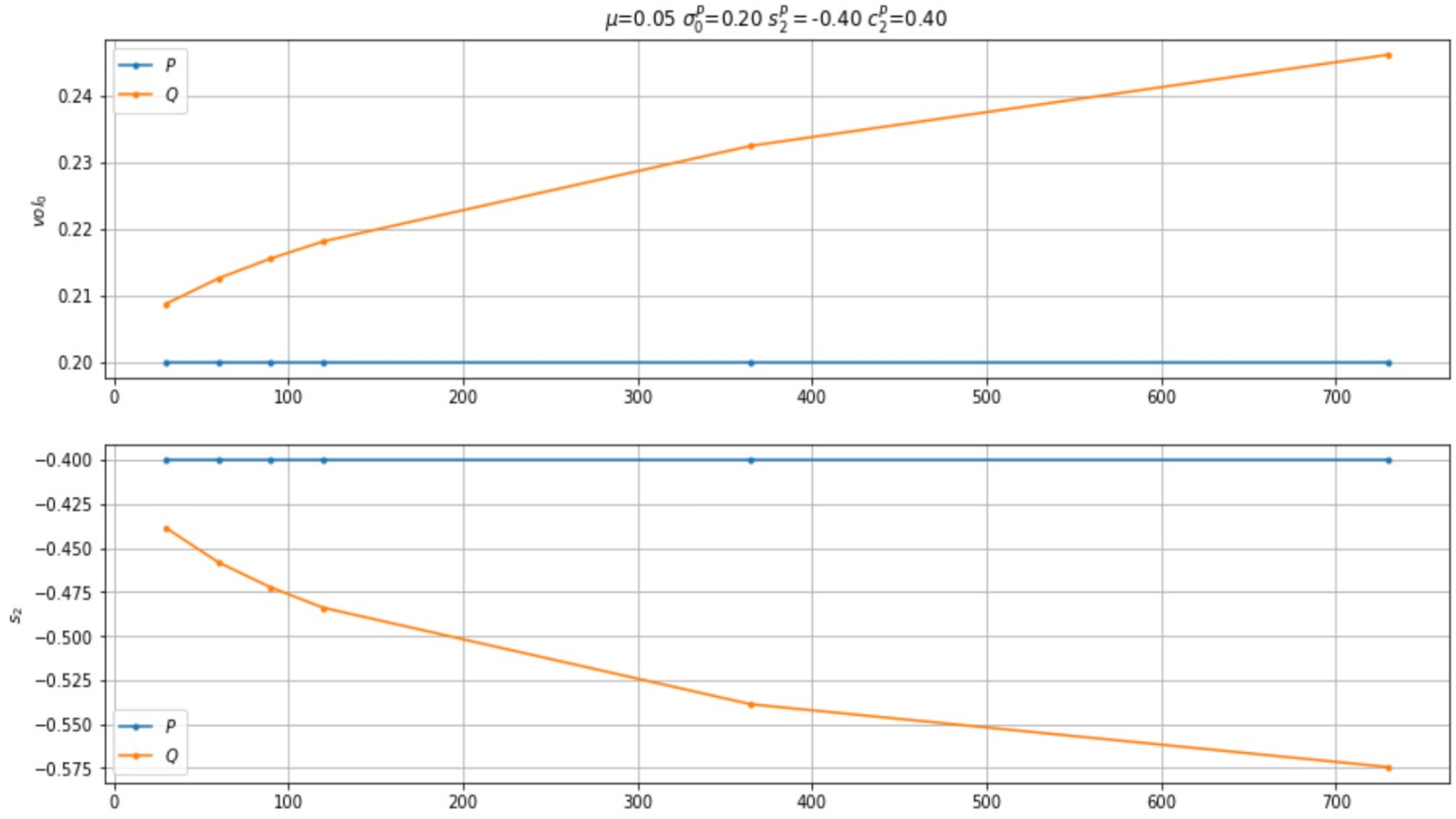


Equity risk premium leads to the existence of vol and skew risk premium!

All we assumed:

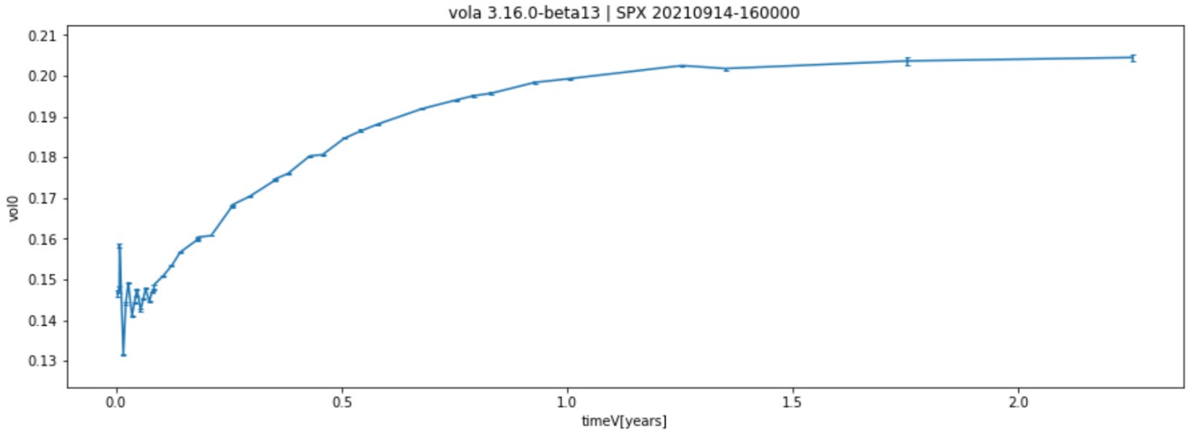
- Investing in stocks makes money
- Traders optimize average returns

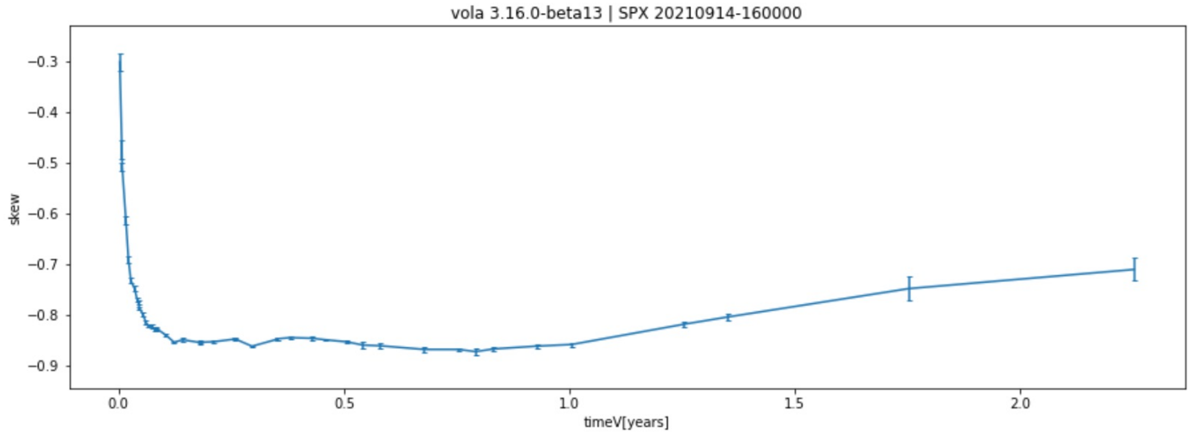
Risk premium term structure



- Volatility risk premium term structure calculated from the equity risk premium qualitatively looks like the one observed in the market and in the same ballpark numerically!
- s2 skew in the market (*Q* measure) usually slowly decays with T. This analysis indicates that it may decay faster under *P* measure.

SPX parameter term structure





Conclusion

- Optimal option portfolio is given by an explicit formula as a ratio of the probability densities under *P* and *Q* measures.
- Firms with alpha in delta and vol space could noticeably improve risk-adjusted returns by using optimal portfolios with options (provided this alpha is not priced in option prices as described here) .
- It is easy to describe these densities with parametric vol curves.
- No free lunch in the presence of profit opportunity: one should not be able to increase returns for free.
- Equity, volatility, and skew risk premia are not independent. Given the probability distribution under *P* one can uniquely determine *Q* measure and the corresponding implied vol surface.
- The resulting term structure of vol risk premium is consistent with the shapes observed in the market and has the right ballpark values.