State of the Smile: The Ever-Surprising Evolution of the Equity/Listed Options Market

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Vol Skews: 2008 versus 2020

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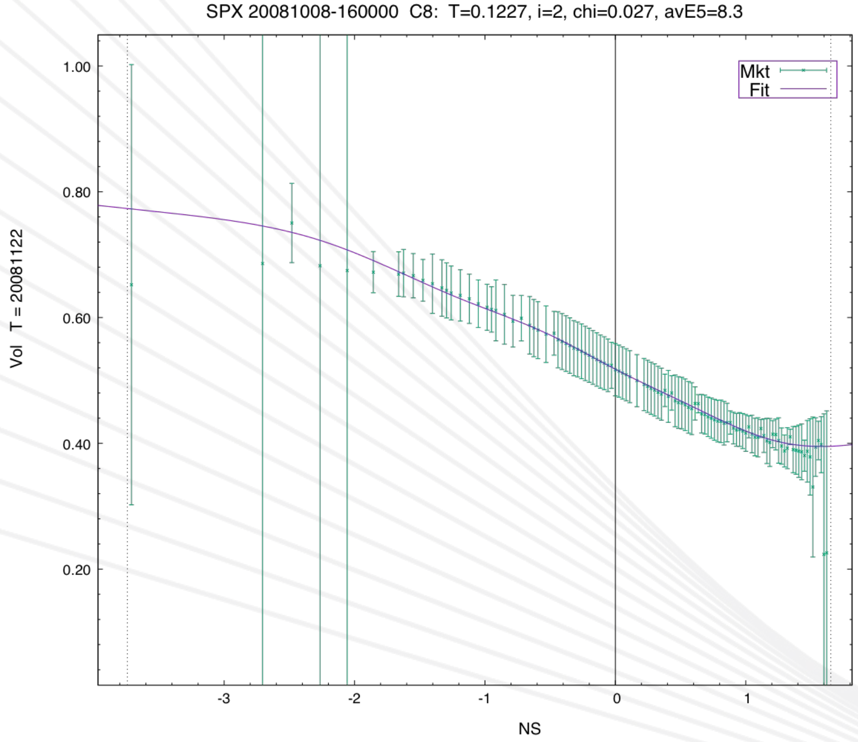
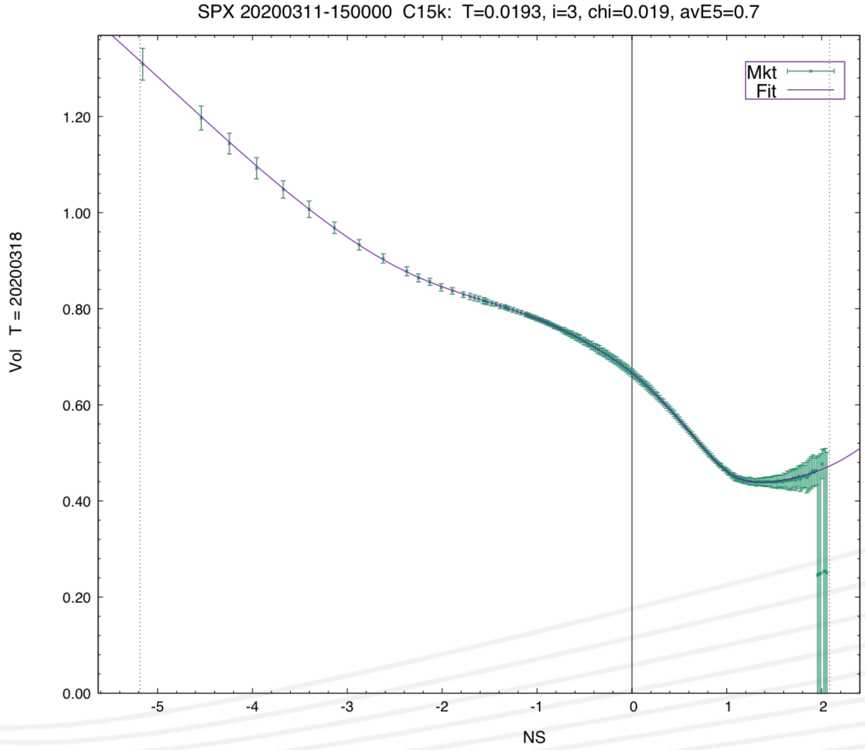
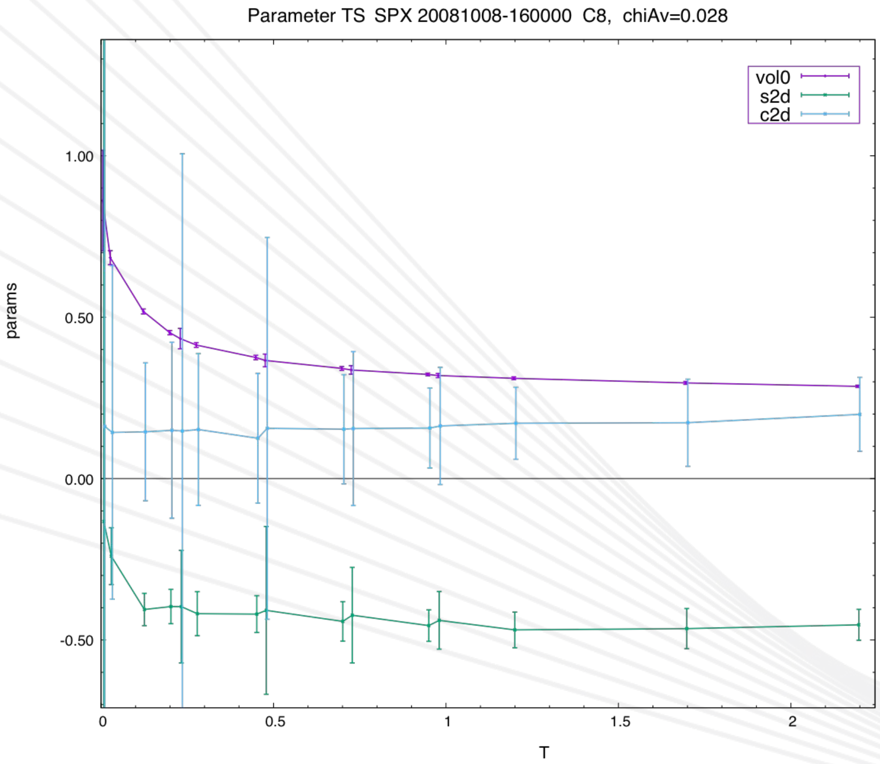
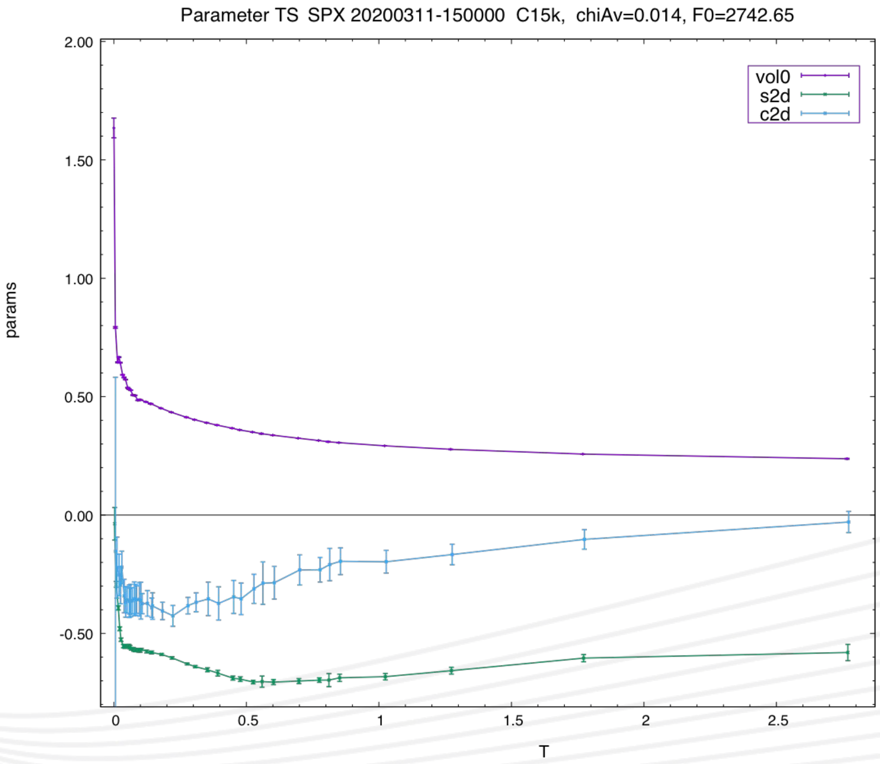
Analytics for Options Trading





Parameter TS: 2008 versus 2020





Introduction

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- The listed options market has grown dramatically over the last two decades.
- Prop shops and hedge funds are much more important players now.
- OTC flow and exotics markets can't ignore the listed (vanilla) market (but still try?)
- The events of the last 15 years have created or brought to the surface new facets of the market one has to consider.
- The two main threads of this talk are:
  - The listed options market has become very "sophisticated and opinionated". It contains a lot of useful information.
  - All the (modeling and algorithmic) details one has to get right to create and maintain a large-scale valuation infrastructure.

Listed (mostly Equity) Options Markets Overview

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- In US alone: circa 1,600,000 options on 5,600 underliers (OPRA, Oct 2023)
  - SPX has about 20,000 options (calls and puts) and about 60 expiries these days!
- Vanilla valuation is complicated due to dividends, borrow costs, rate term-structure, events, settlement/calendar details, vol-time, and vol curves with lots of structure.
  - American "vanillas" are really exotics!
- OMM: All options can only be valued with real-time, robust implied borrow curves and well-designed & calibrated implied volatility surfaces.
  - Also required for real-time risk, PnL decomposition, margin, exotics, etc.
- All borrow and vol curve modeling and fitting analytics etc are proprietary.
- Low latency / HFT puts a lot of pressure on quant models and algos (esp. for OMMs)!

Implied Vols and Surfaces

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- Implied volatility surfaces (& borrow/forward curves) are the standard approach to summarizing the vanilla options market in an intuitive and compact manner.
- They provide the fundamental building block for the trading of vanillas (listed and OTC), as well as flow derivatives and exotics.
- There are many quant problems facing options and derivatives trading desks, and the problem of constructing sensible, arbitrage-free volatility surfaces from options market prices (bids and asks) is one of the hardest.
- This issue already exist for European-style options (SPX, SX5E, DAX, etc).

Implied Vols and Surfaces (cont'd)

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- For European options (without divs) only integrated rates and variances matter.
  - Cash dividend modeling is relatively minor issue for Euro options (unless stochastic divs…).
- But American options are really path-dependent exotics and a lot of extra complications arise (esp. for ETFs, stocks, esp. with dividends):
  - Need to choose proper cash dividend and borrow cost modeling. Then:
  - Even in BS: Besides rate term-structure, proper choice of "vol time" (aka "business time"), including "event time" affects early exercise premia, and all details matter, incl. "settlement".
  - Beyond BS: Local vol? Stochastic LV? (Look at volga, vanna…)
  - Approxs/hacks to adjust ITM relative to OTM vol to still price call & put of same strike in BSM.
- There are subtleties in "De-Americanization" (see above and at the end…), but if in doubt think of "implied vol surfaces" as summarizing European options prices in a convenient and intuitive manner (whether they are listed/traded or not).

Vol Surface Parametrizations

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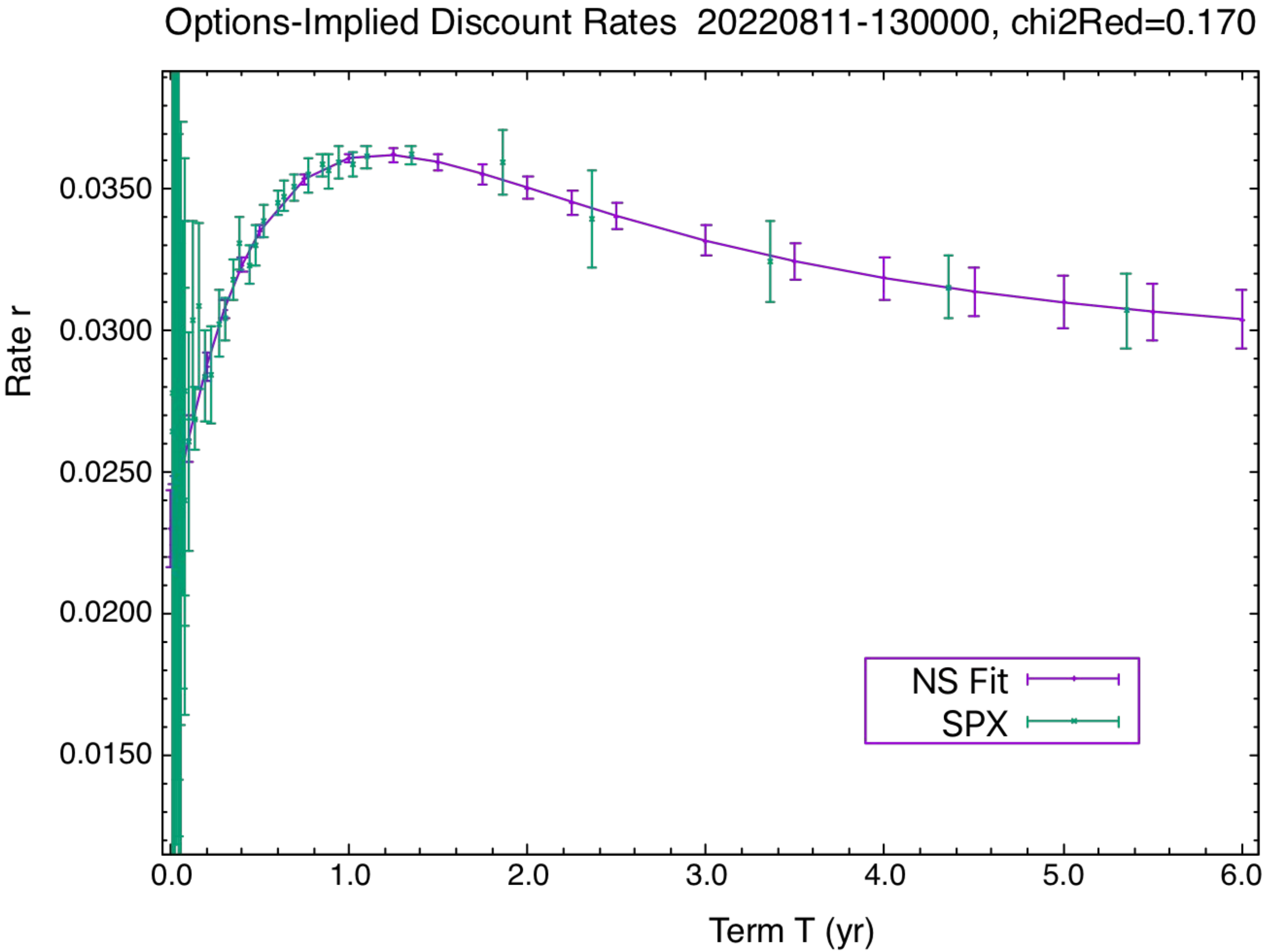
- There are of advantages to having a good vol curve parametrization per term:
  - Intuitive parameters, as independent as possible, stable from fit to fit.
  - Smooth* (in strike) over regions that are strongly correlated (cross-hedging).
  - Comparable across terms, little term-structure if possible (except small T perhaps).
  - Makes it “easy” (easier…) to avoid arbitrage, e.g. Lee bounds can be built in.
  - Allows easy scenario generation, finding market opportunities, etc.
  - Easier to design an auditable and (human-)adjustable large-scale infrastructure.
  - Give fast and robust local vols, and help with other exotics model calibration issues.
- A parametrization of the term-structure is not as crucial (it’s also very hard):
  - Good curves are easy to interp/extrapolate in T — but tie together to avoid calendar arb!
  - Dupire formula is 1st order in T, 2nd order in strike…

Beyond S* curves: C* curves

- Liquid names can not be fit with simple public-domain curves like S3/SSVI, S5/SVI, SABR (**S* curves**), or parabolas, etc:
    - S* curves have a unique, positive maximum in their curvature around ATF, **c2 > 0**.
    - Note e.g. that any kind of **event** (earnings, elections, Brexit, covid, etc) can lead to **bi- or multi-modal distributions**, which generally require **c2 < 0**.
    - This is true not just for equity, but **also for FX, IR**.
- Need curves that allow more general curvature structures, including **c2 < 0**, but can be made arbitrage-free and fitted robustly and fast.
- Vola Dynamics designed such curves: **C* curves**: C5, C6, C7, ..., C16
- Details later. First...

"SPIBOR" — Even the Fed cares now!

- What discount curve should you use for your options trading?
- Depends... but for implying borrows, vols, etc, use market consensus.
- Euro PCP for given term T: $C - P = DF\ F - DF\ K$
- For each disc factor DF(T) need a robust linear regression across many strikes K.
- For further robustness, can smooth rates via a term-structure fit.
- Why does the Fed care?
  - Treasuries, SOFR, etc are NOT risk-free rates!
  - They can be lower than risk-free ("convenience yield"), or higher ("default risk").
  - Usually lower, by 20 – 40 bps (almost flat).
  - SPX options MMs should be using close to risk-free rates ("box rates") due to margin requirements at exchange and OCC level.



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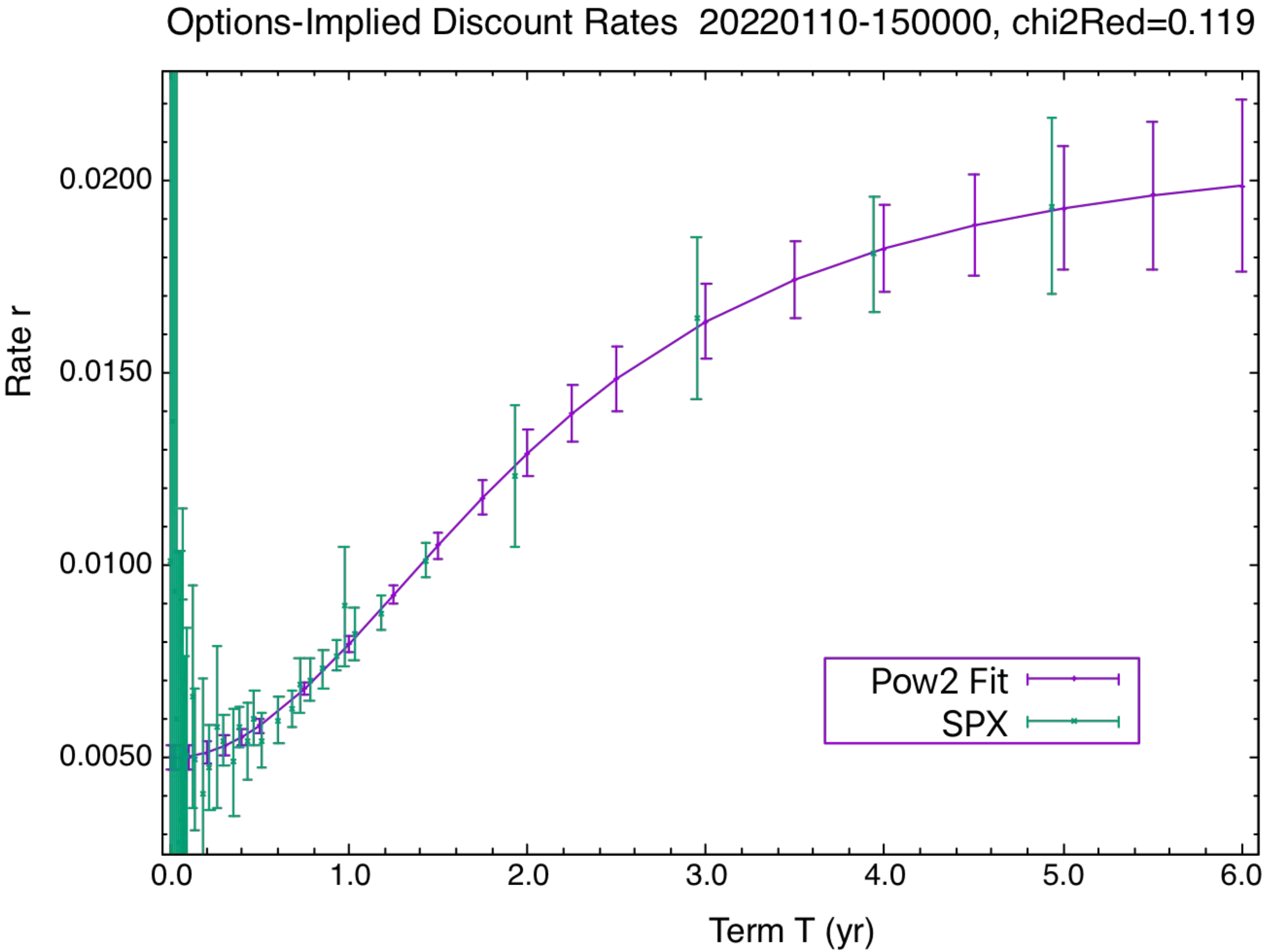
What discount rates should I use?

SPIBOR

Just one snapshot!

Nelson-Siegel TS fit



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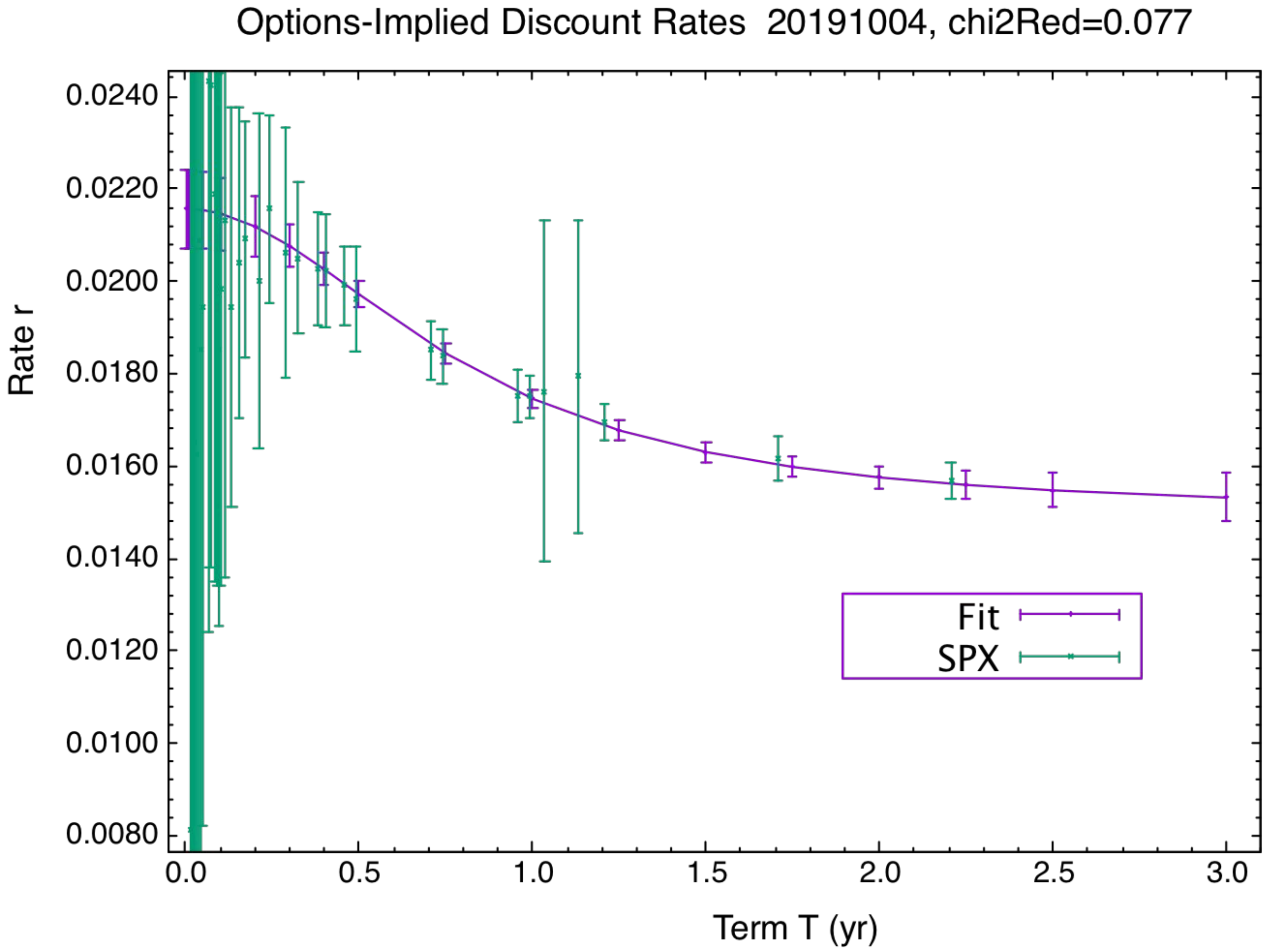
What discount rates should I use?

SPIBOR



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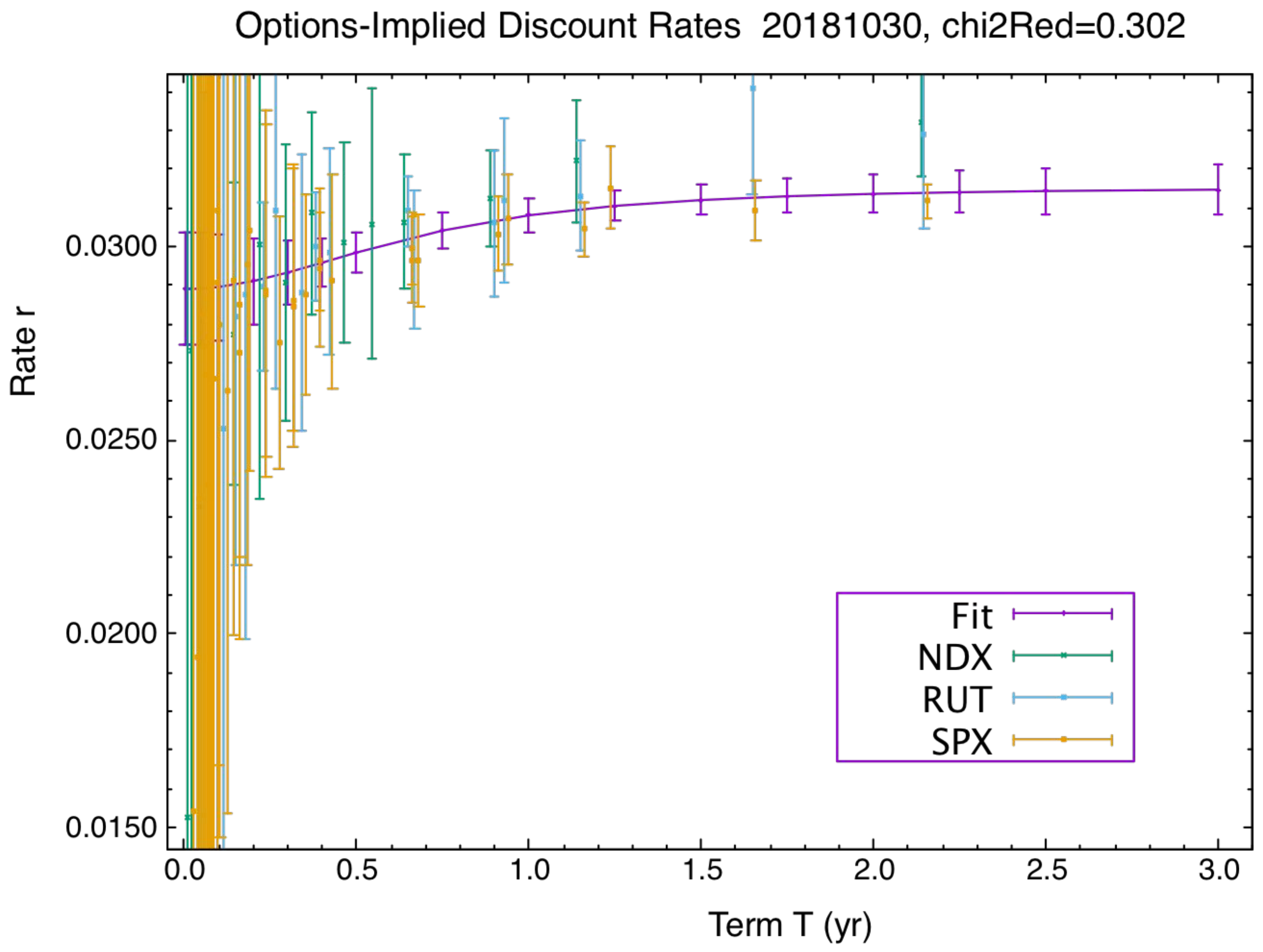

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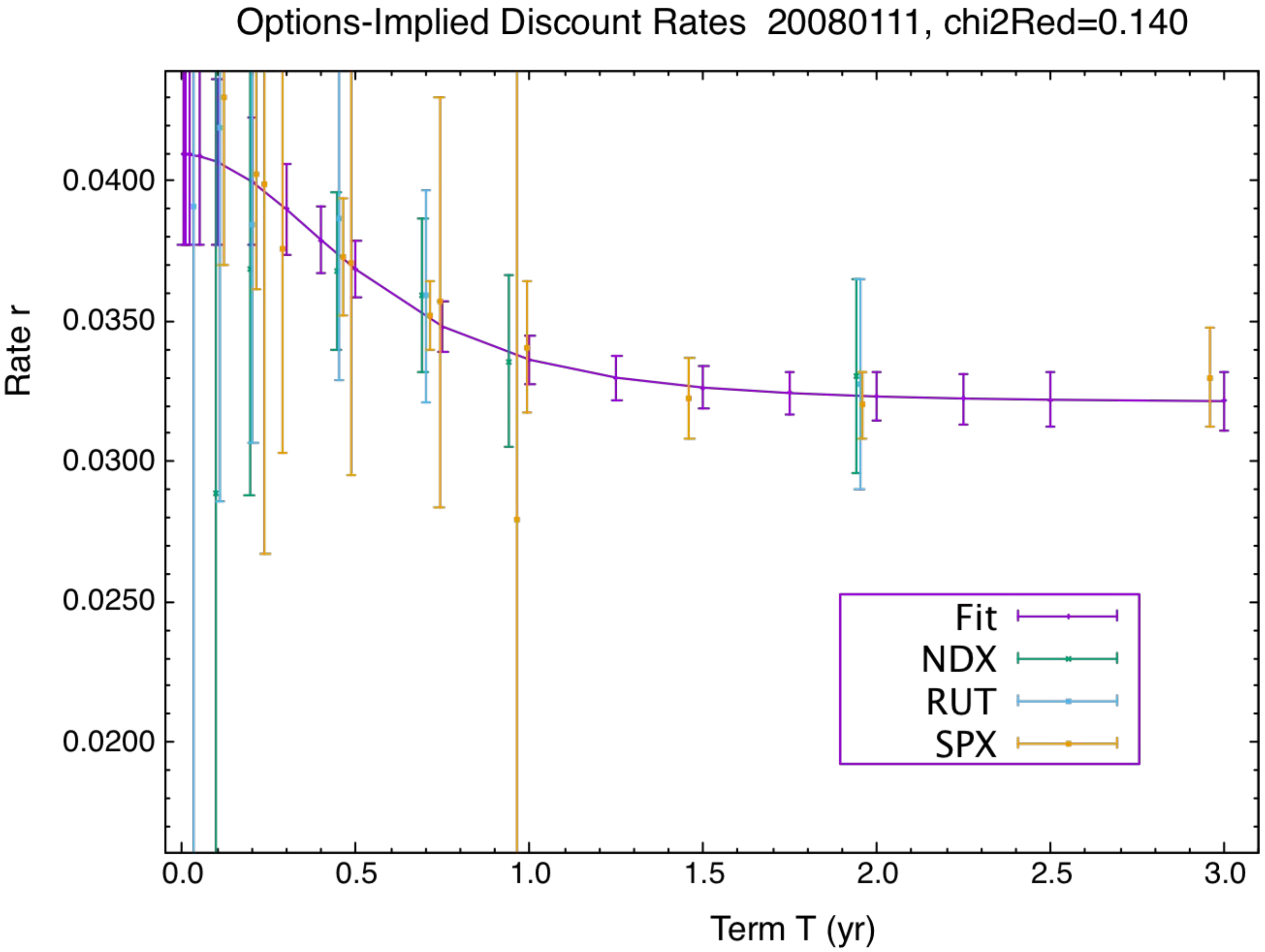
What discount rates should I use?

Maybe they are underlier/sector dependent?

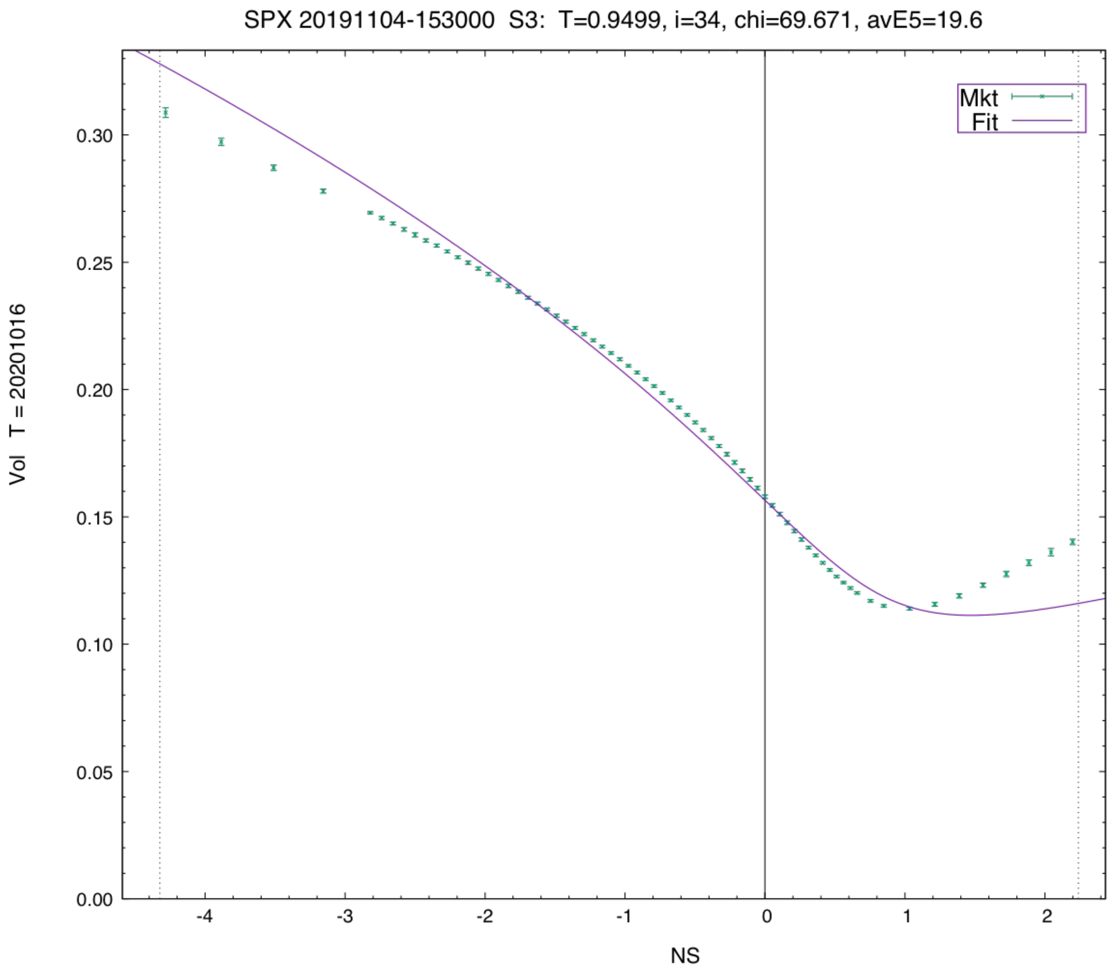


What discount rates should I use in 2008 ??

SPIBOR

Vol Fitting Examples

- Given disc rate and divs, we first imply borrows or forwards (BS vs Black...)
- When implying vols we "de-Americanize" the options if needed...
- We then fit implied vols to suitable vol curves in each term, while transferring info across terms to avoid cal arb, etc.
- So we're purely concerned with the vol fitting problem here (not EEP).
- We will show in each plot:
  - **Curve type:** S* (S3/SSVI, S5/SVI), C* (C5, C6, C7, ... , C16) with #params.
  - **chi** aka chi2Reds: Standard relative (to "error bars") quality-of-fit metric (statistical).
  - **avE5** aka avgErrors5: Average of the absolute difference between fit and market implied vols for 5 strikes around ATM (in bps).



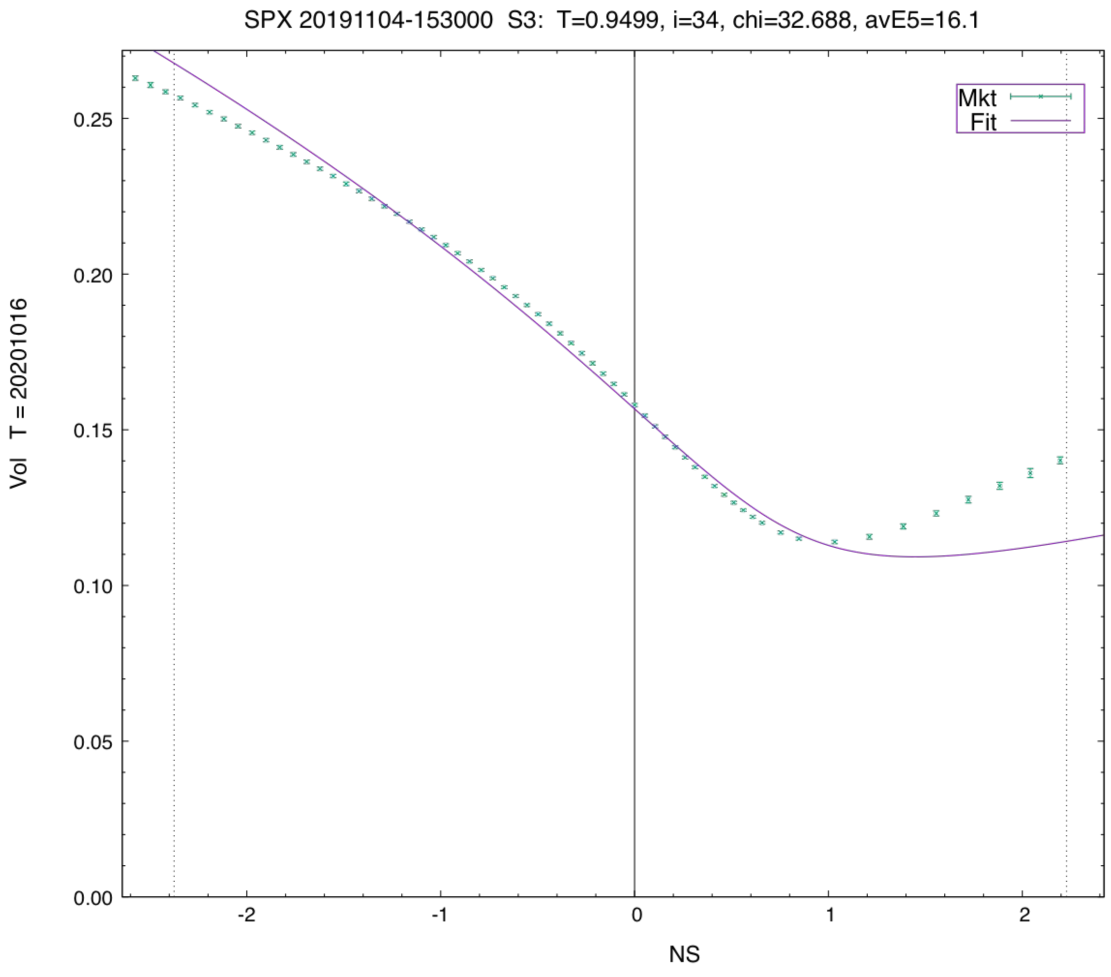
Can non-W shapes be fitted with simple curves?
For large terms at least?

SPX 20191104

SSVI / S3 fit, i=34, T=0.95y

This is a lousy fit even over a medium range…

$$z := \mathrm{NS} := \frac{\ln(K/F)}{\sigma_0\sqrt{T}}$$



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Analytics for Options Trading

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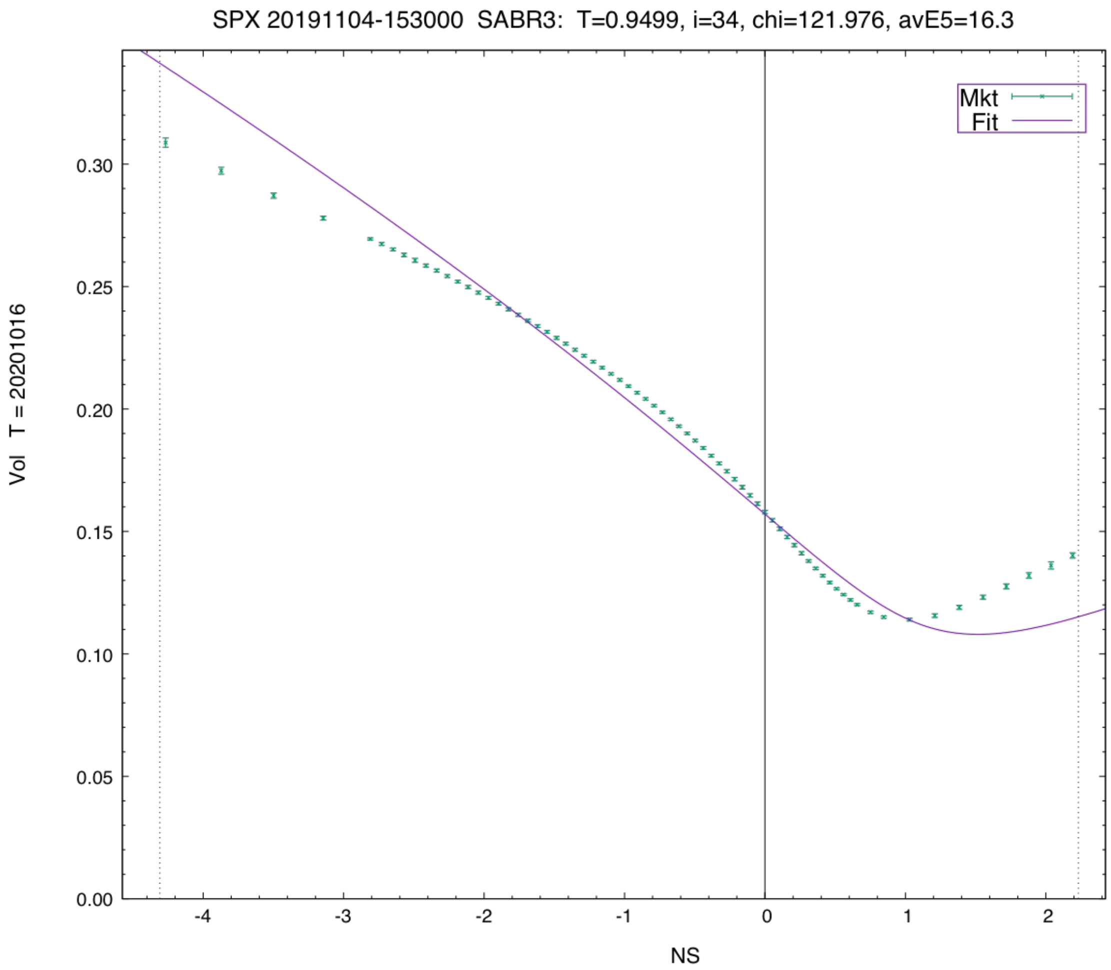
For large terms at least?

SPX 20191104

SSVI / S3 fit, i=34, T=0.95y

This is a lousy fit even over a small range…

… even though shape looks "simple" (c2>0) and this is a supposedly easier longer T.



VOLA DYNAMICS
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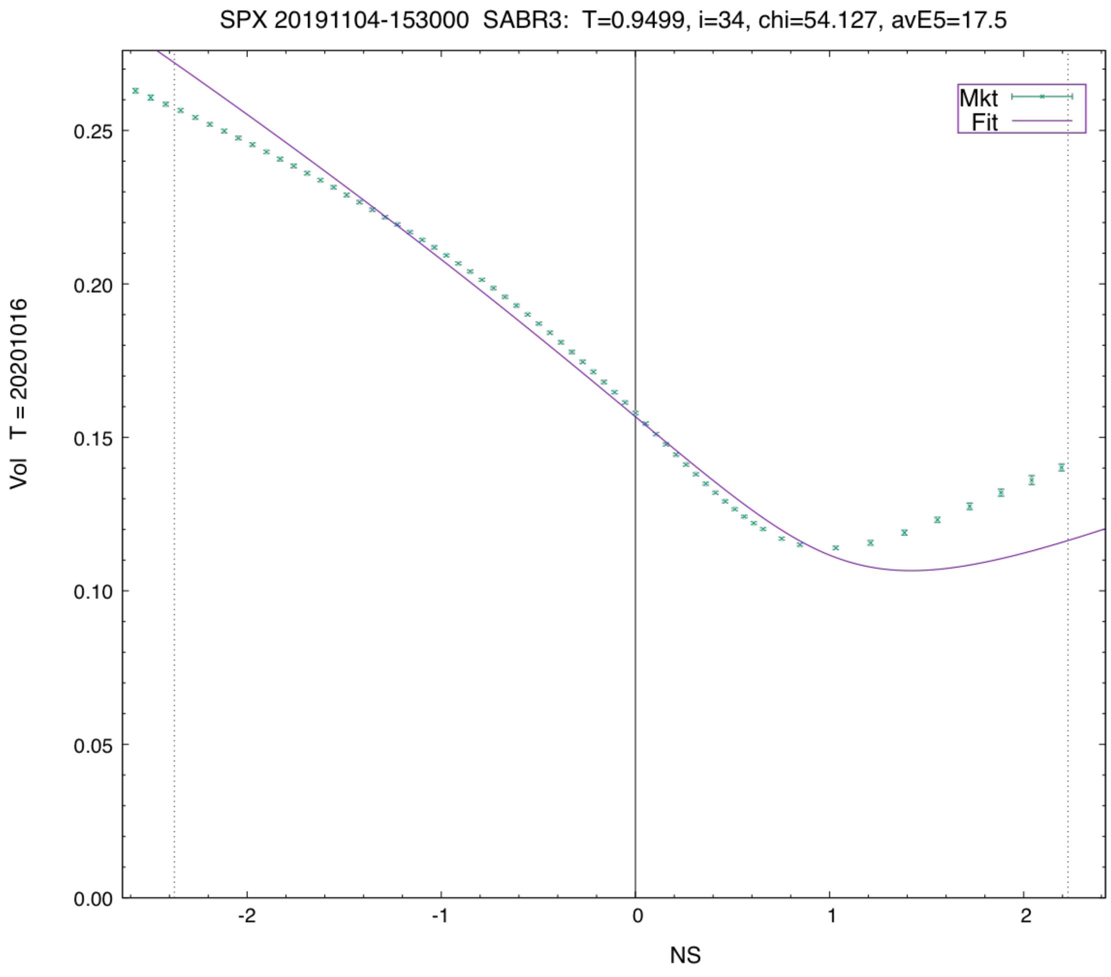
SPX 20191104

SABR fit, i=34, T=0.95y

This is a lousy fit even over a medium range…



SPX 20191104

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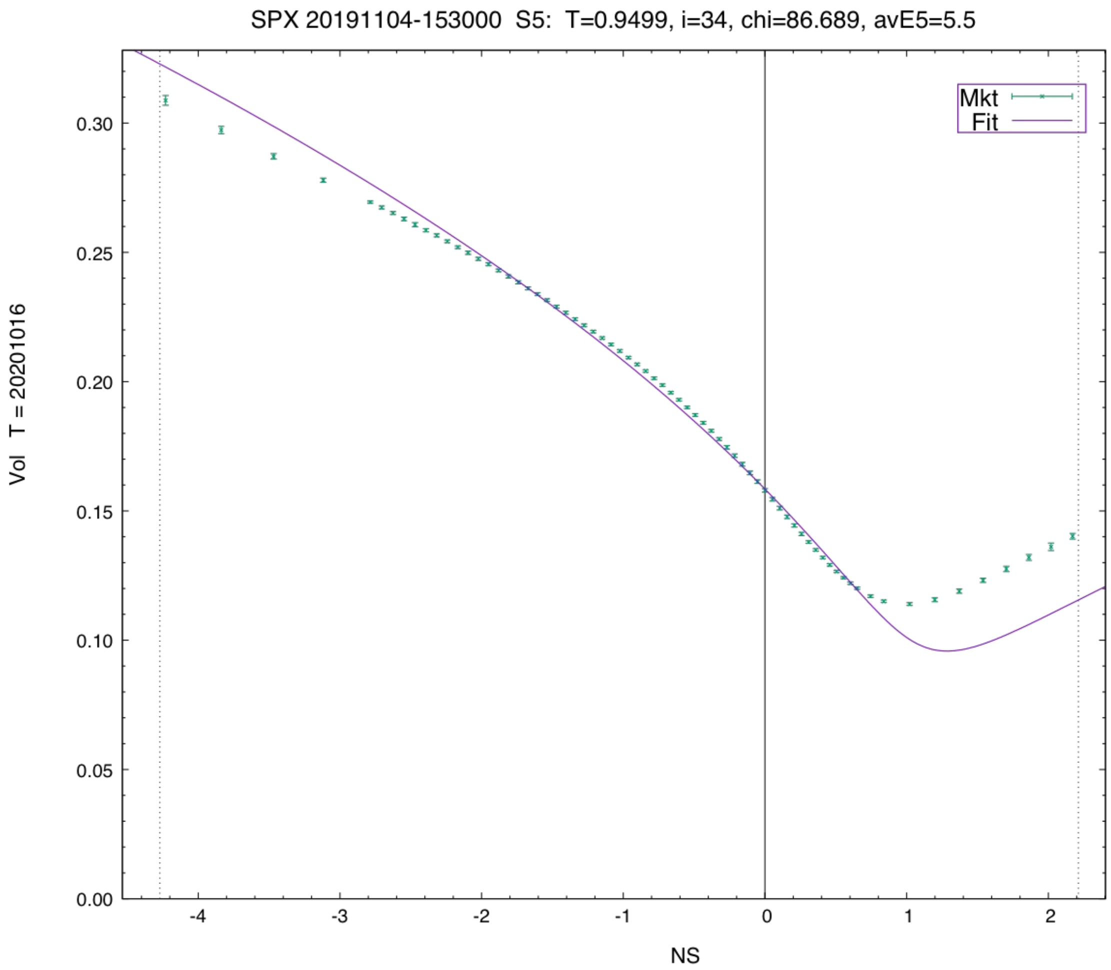

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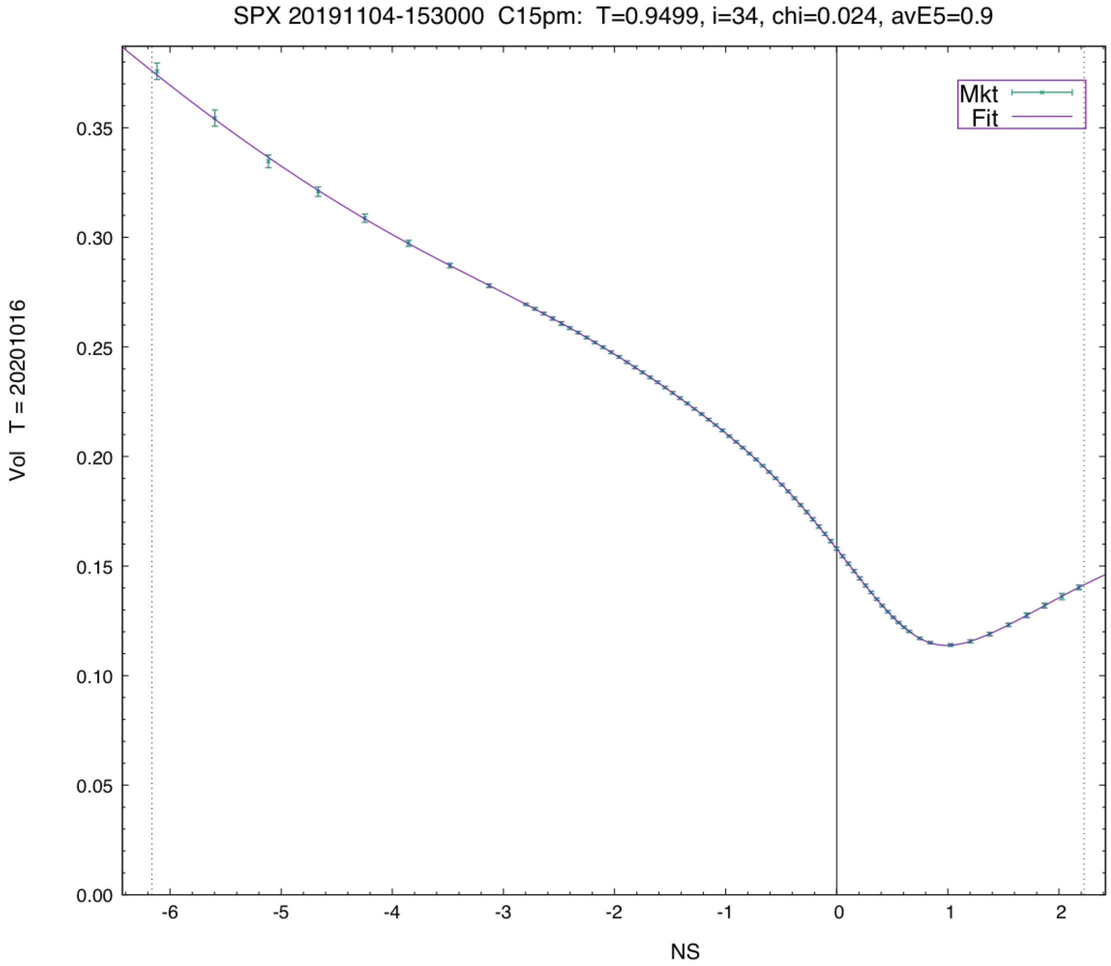
SPX 20191104

SVI / S5 fit, i=34, T = 0.95y

This is still a lousy fit even over a medium range…

Ditto for T = 2y ….



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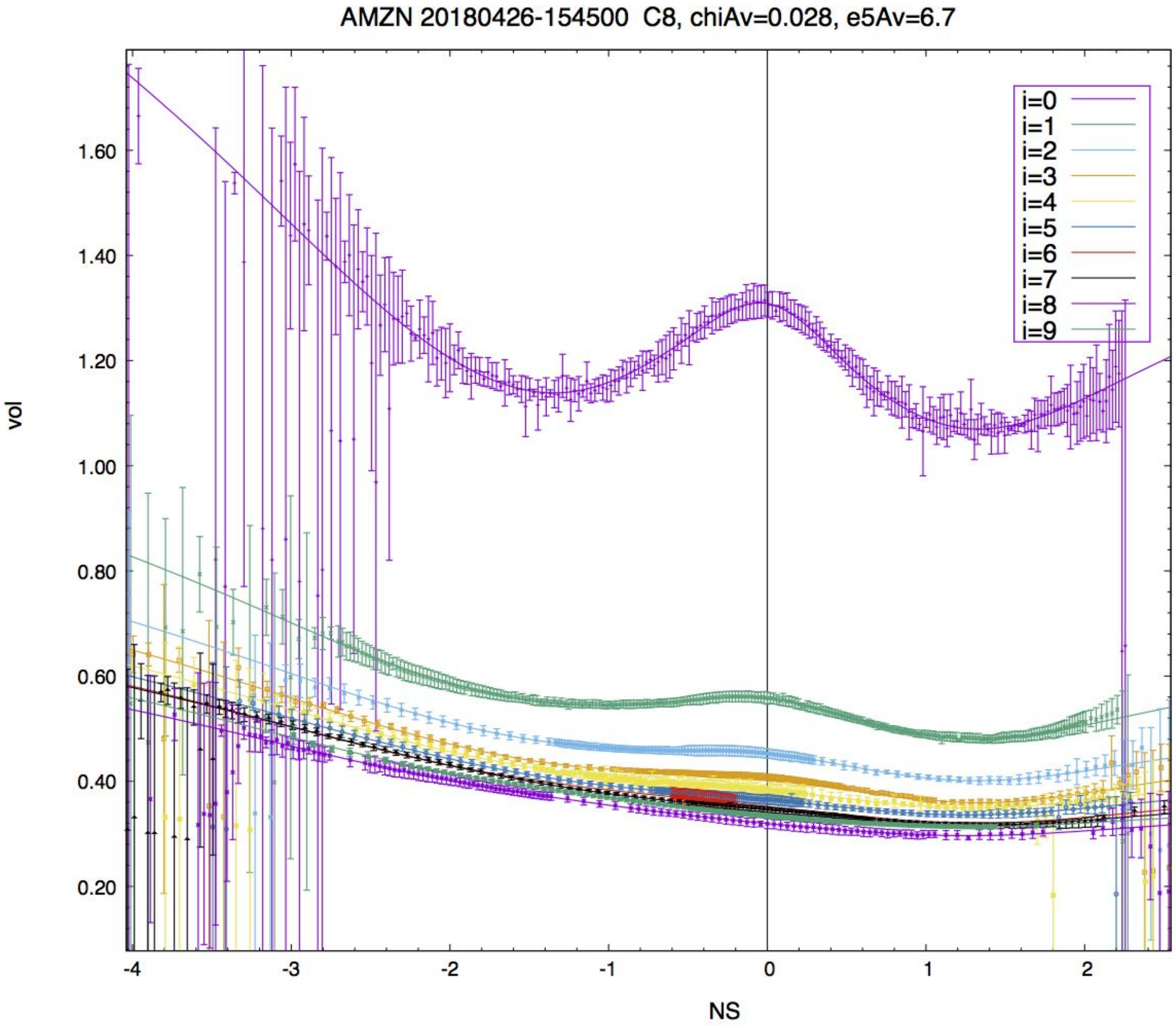
SPX 20191104

C15 fit, i=34, T = 0.95y

This is a great fit over a wide range, and can't be improved w/o over-fitting

chi2 is 1000 – 5000x smaller!

(Yes, curvature of vol^2 is > 0 ATM...)

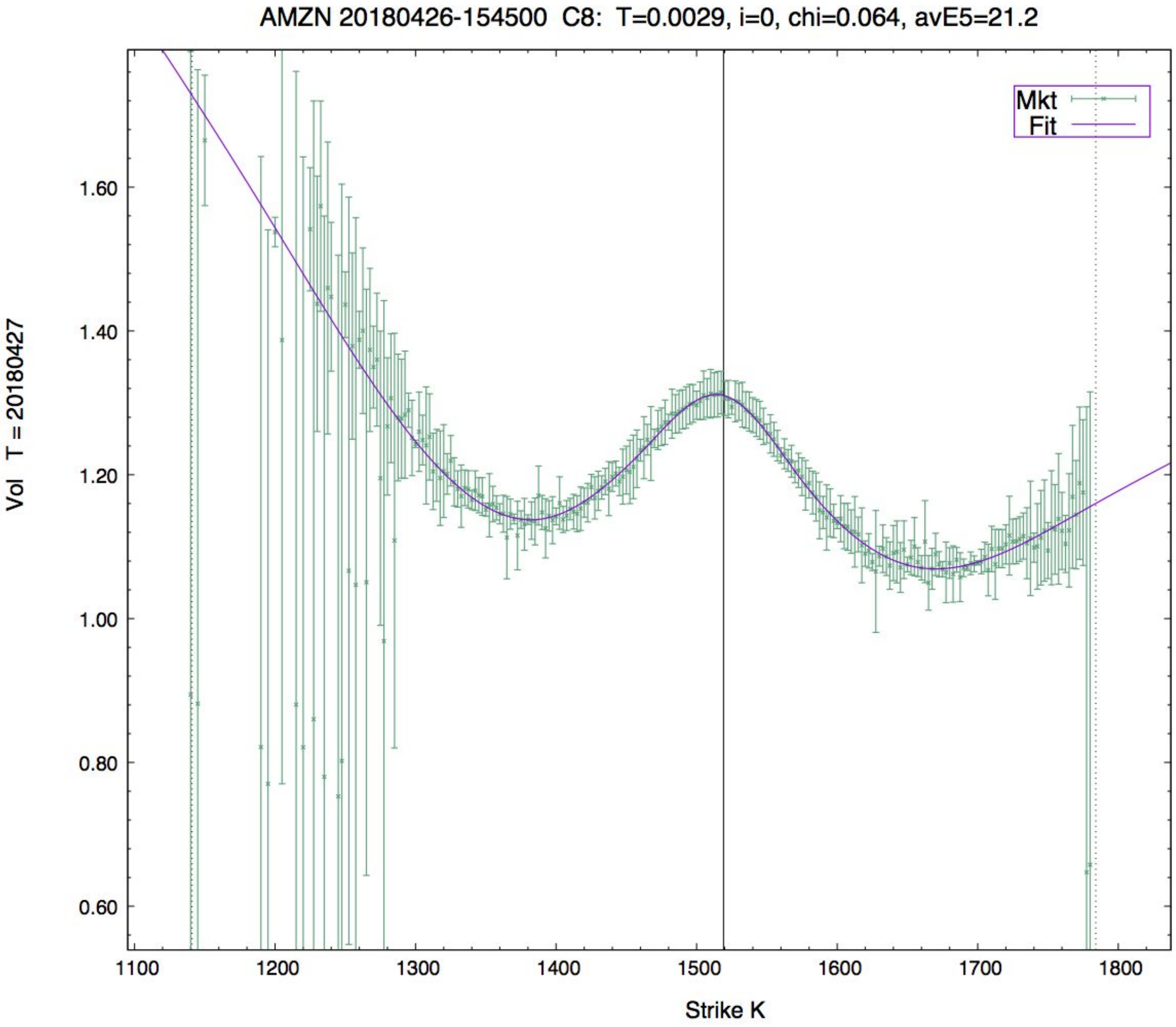


AMZN 2018-04-26 earnings day

C8 Vol vs NS

Interesting Thursday: Earnings, new weekly listed, etc.

$$z := \text{NS} := \frac{\ln(K/F)}{\sigma_0\sqrt{T}}$$

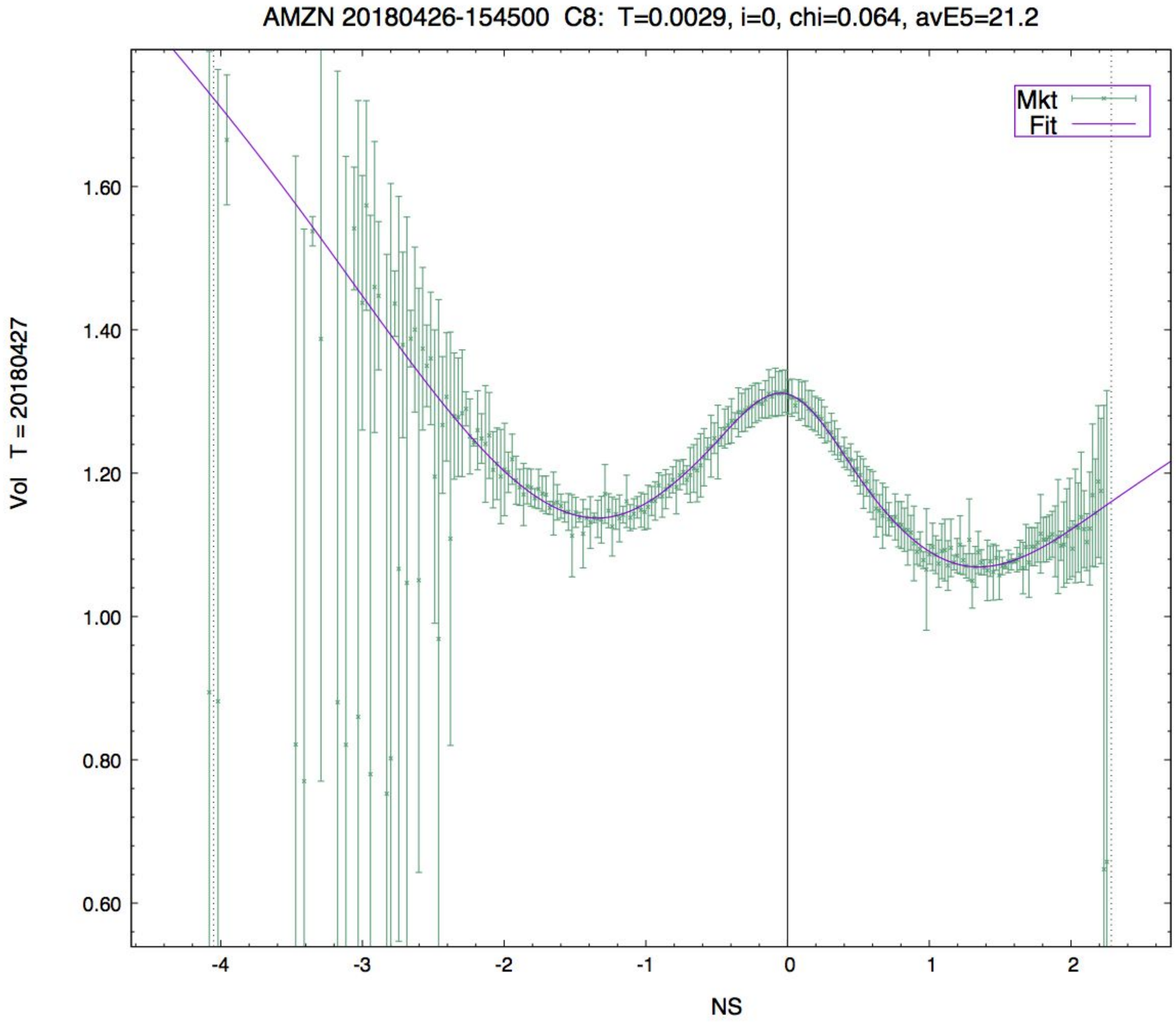


VOLA DYNAMICS
Analytics for Options Trading

AMZN 2018-04-26 earnings day

Vol fit for first term, i=0, K-space

Most negative c2 ever!

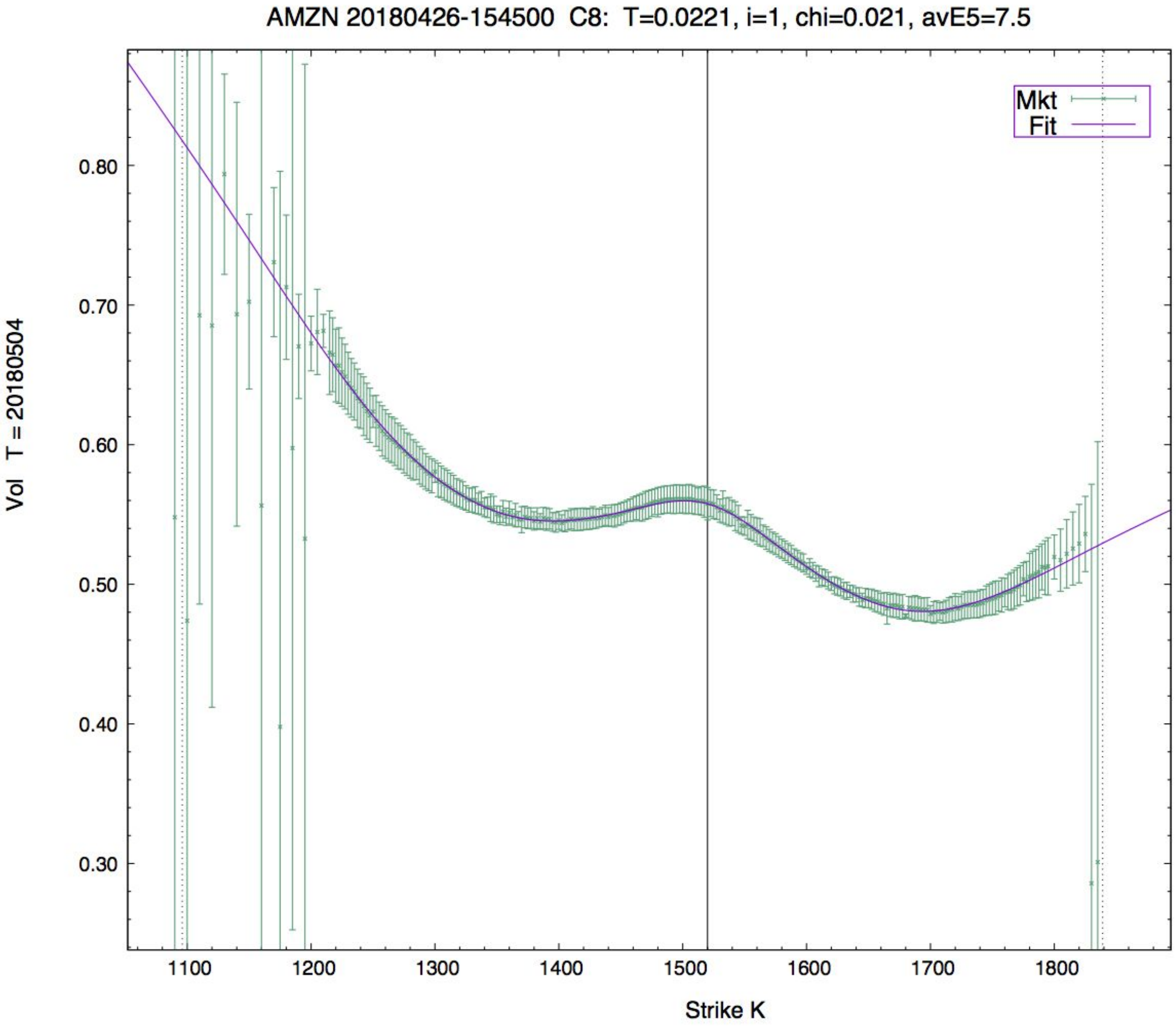


VOLA DYNAMICS
Analytics for Options Trading

AMZN 2018-04-26
earnings day

Vol fit for first term, i=0, NS-space

Most negative c2 ever!



VOLA DYNAMICS
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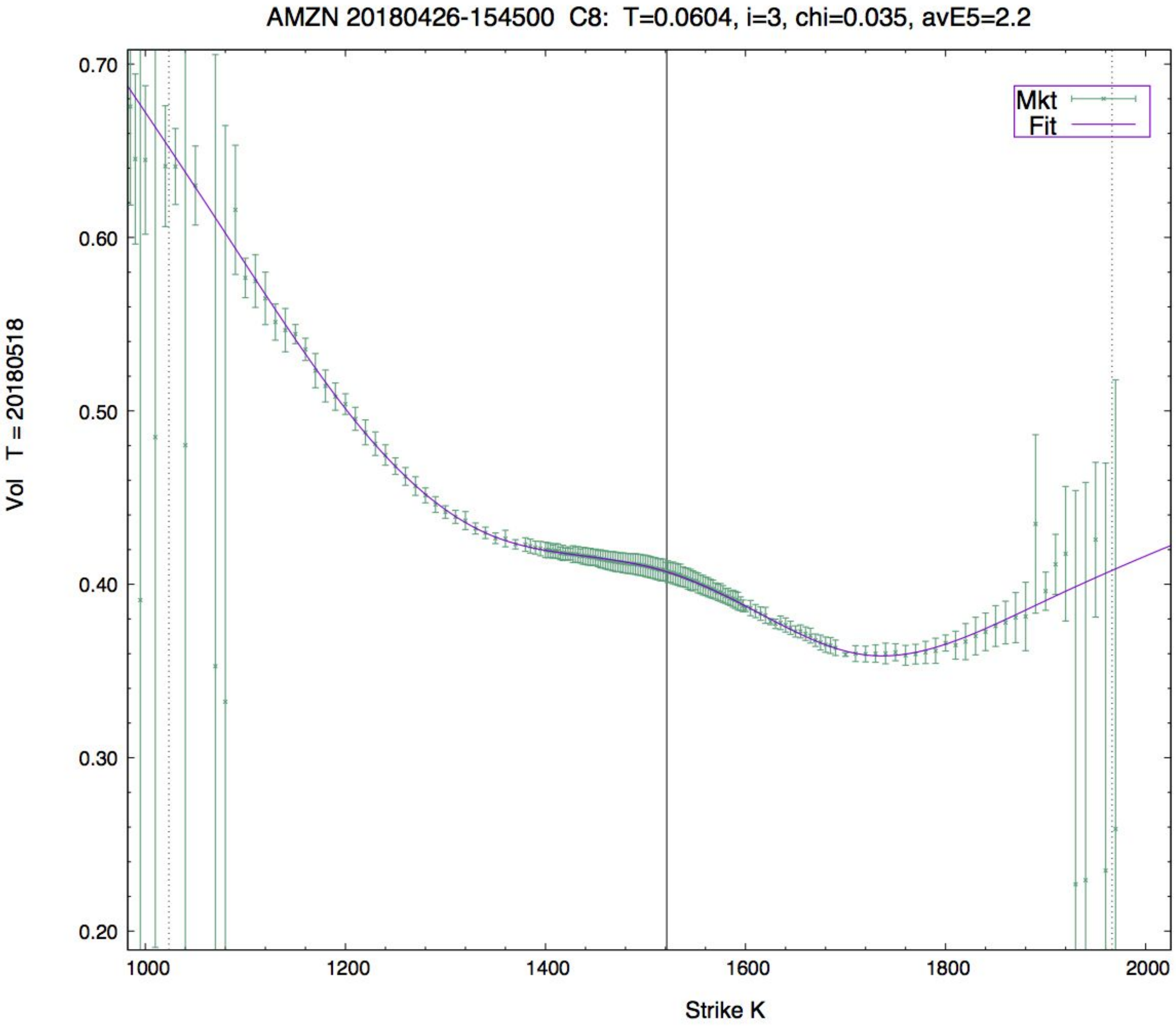
AMZN 2018-04-26
earnings day

Vol fit for 2nd term, i=1, K-space

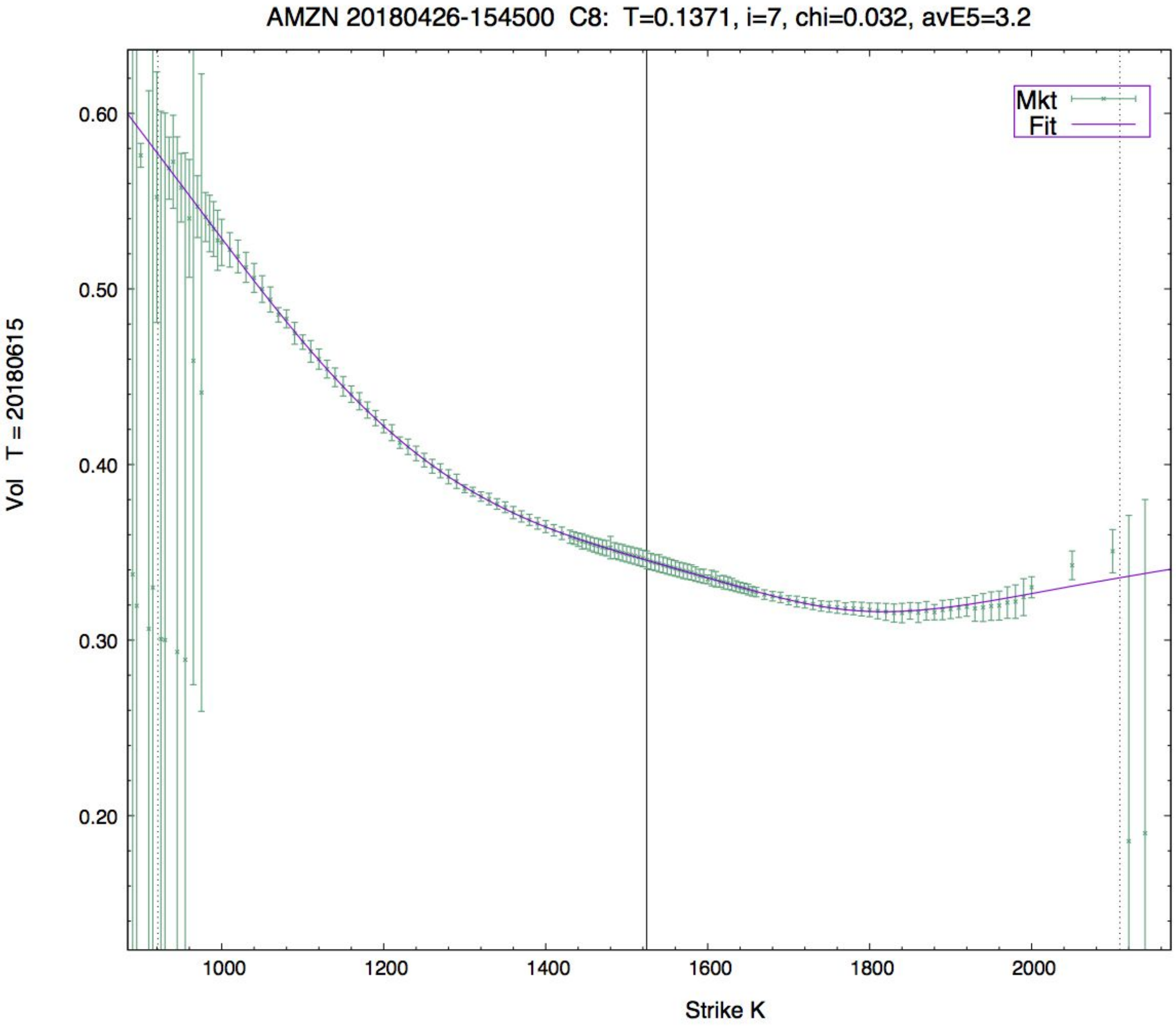


VOLA DYNAMICS
Analytics for Options Trading

AMZN 2018-04-26 earnings day

Vol fit for 4th term, i=3, K-space

Still negative c2!

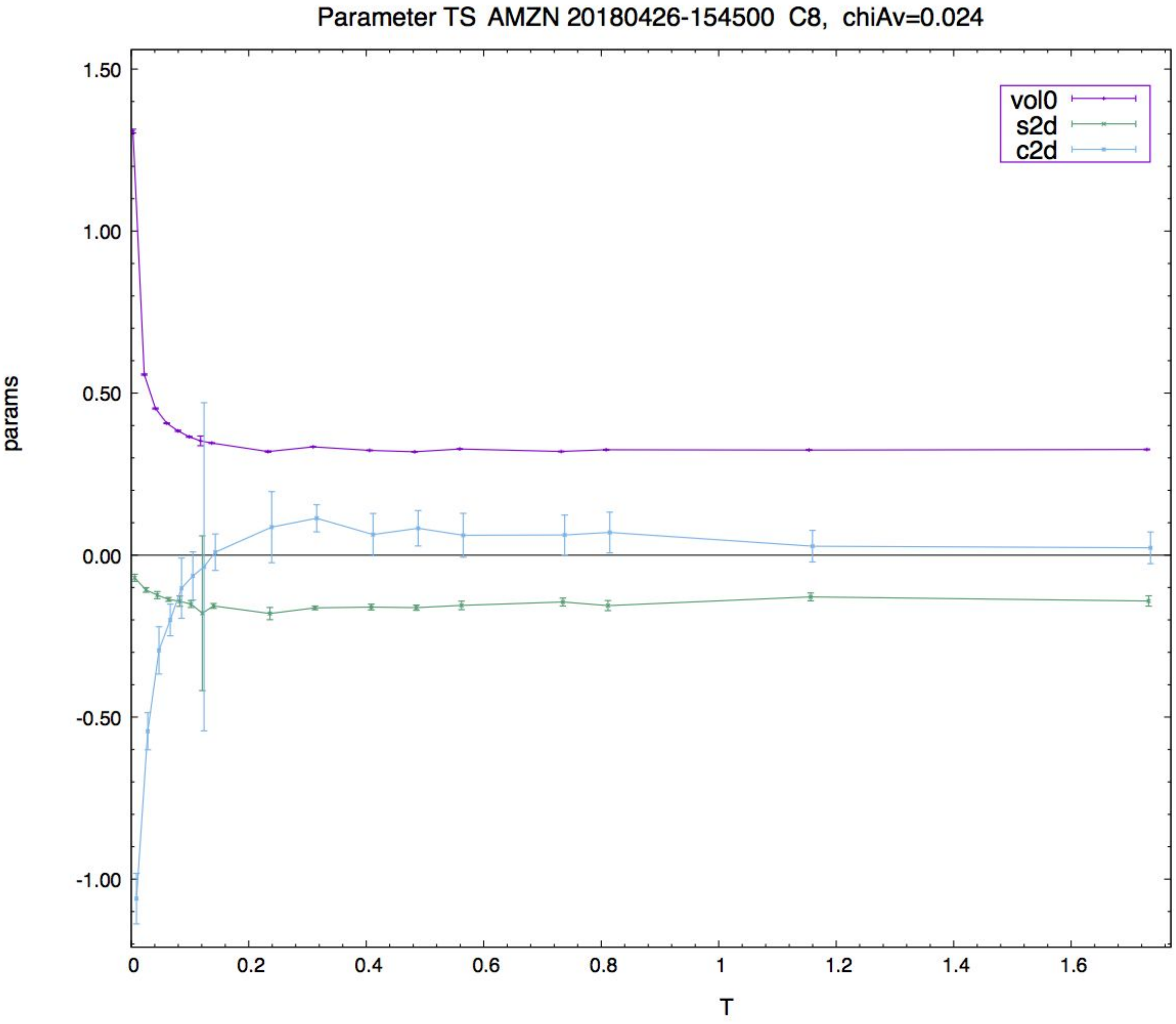


AMZN 2018-04-26
earnings day

Vol fit for 8th term, i=7, K-space

Flat around ATM now, c2≈0.

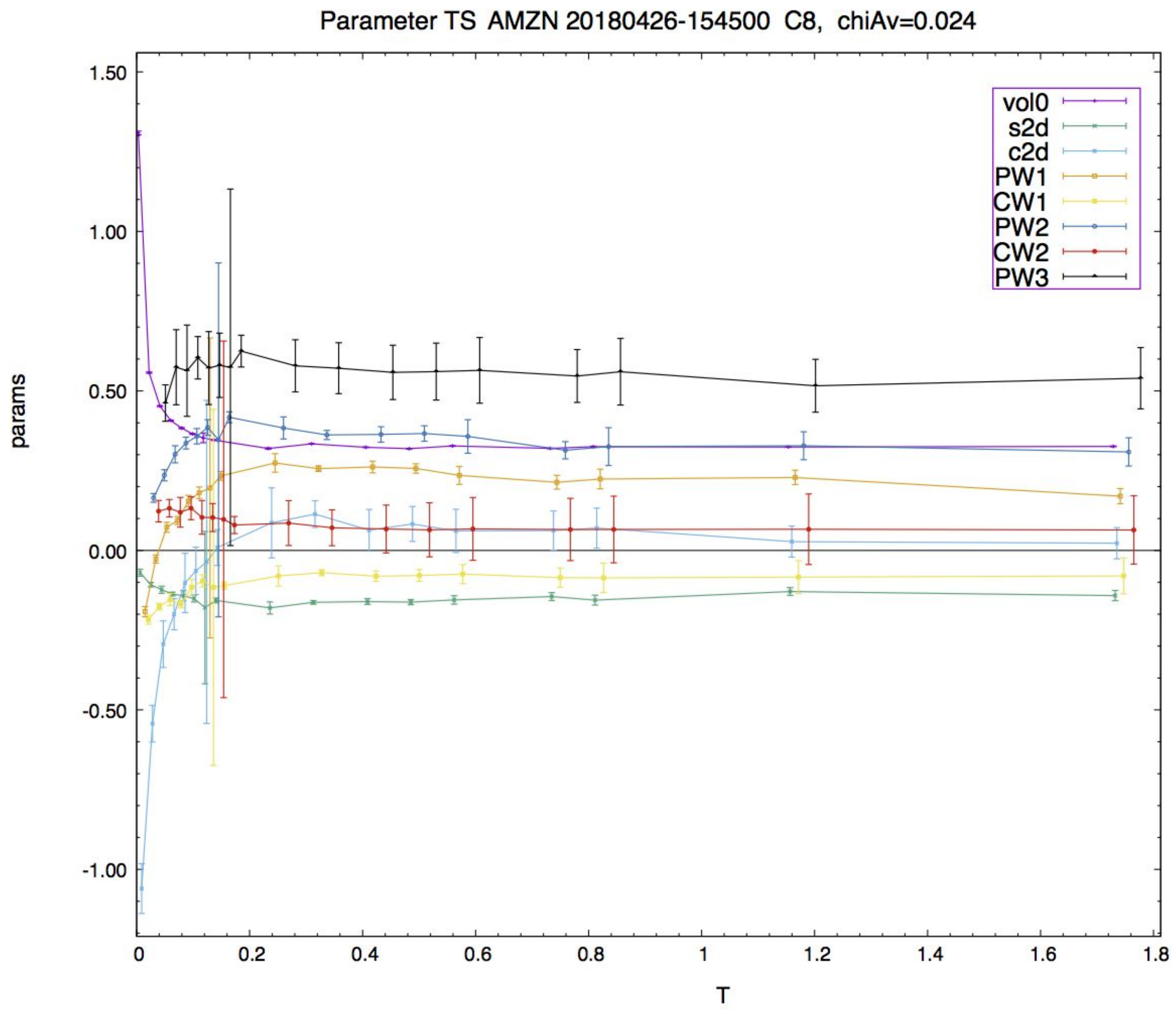
Use C10 if you worry about far wings…



AMZN 2018-04-26
earnings day

C8 parameter term-structure
First 3: vol0, s2, c2

Essentially flat shape params after 3m



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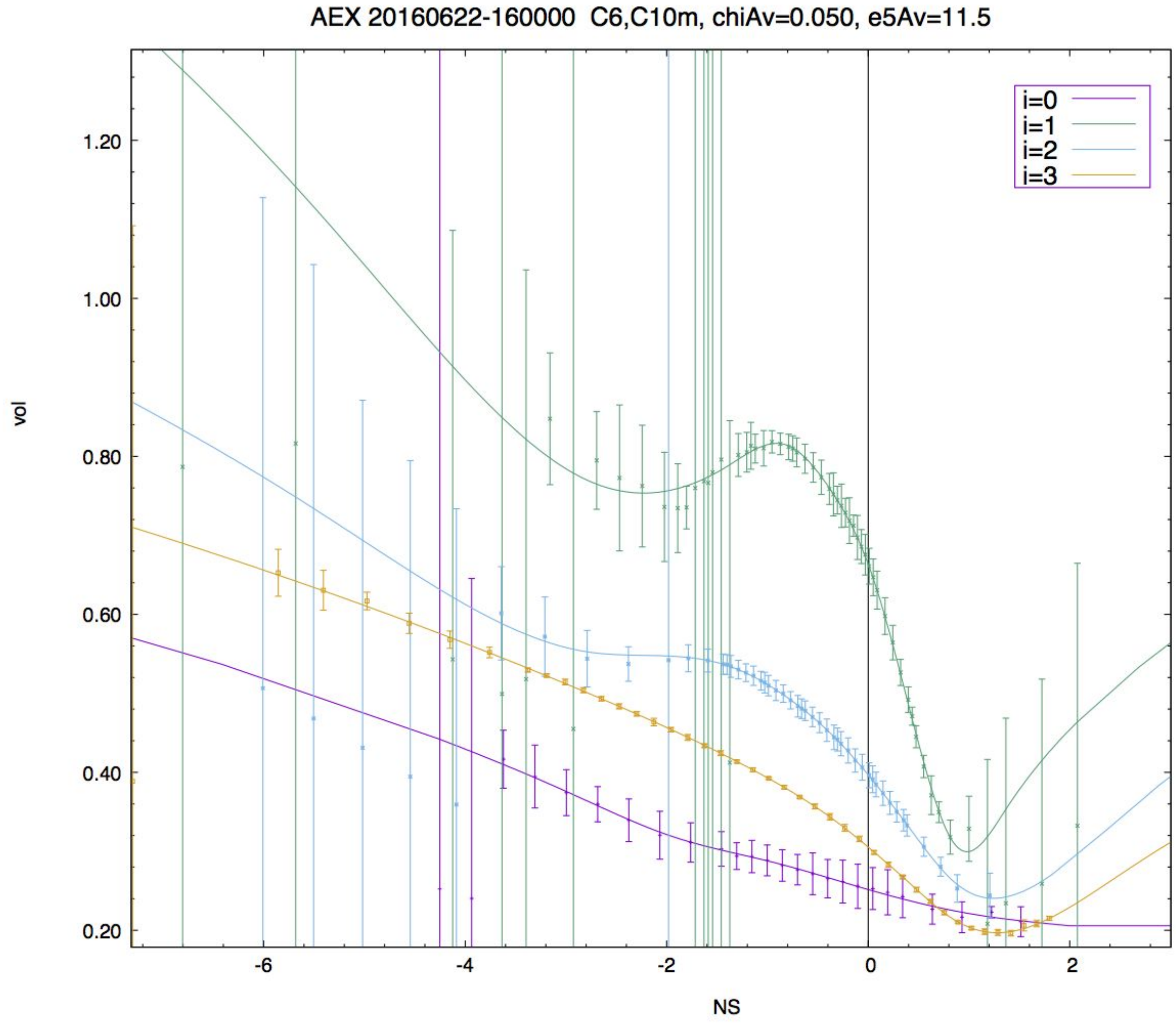

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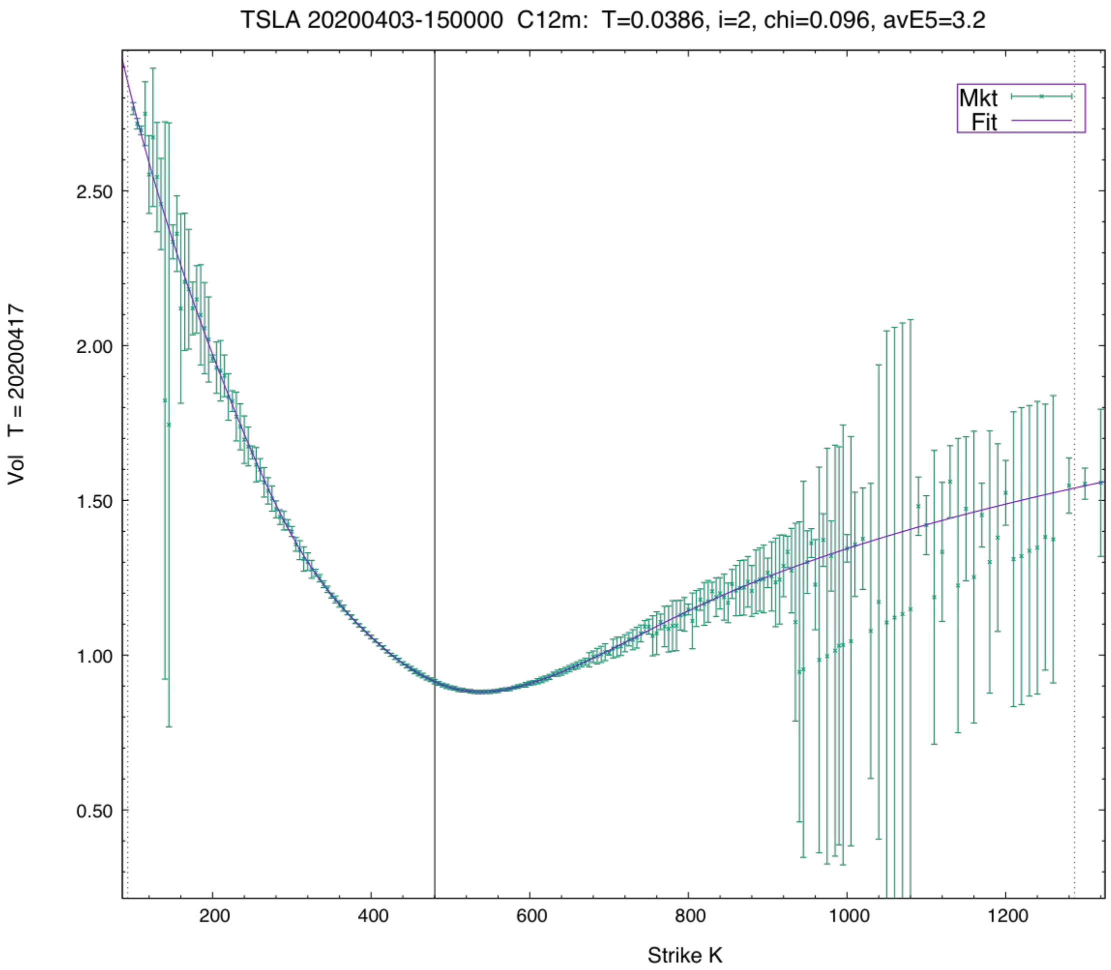
AEX 2016-06-22

Day before Brexit!

Vol vs NS

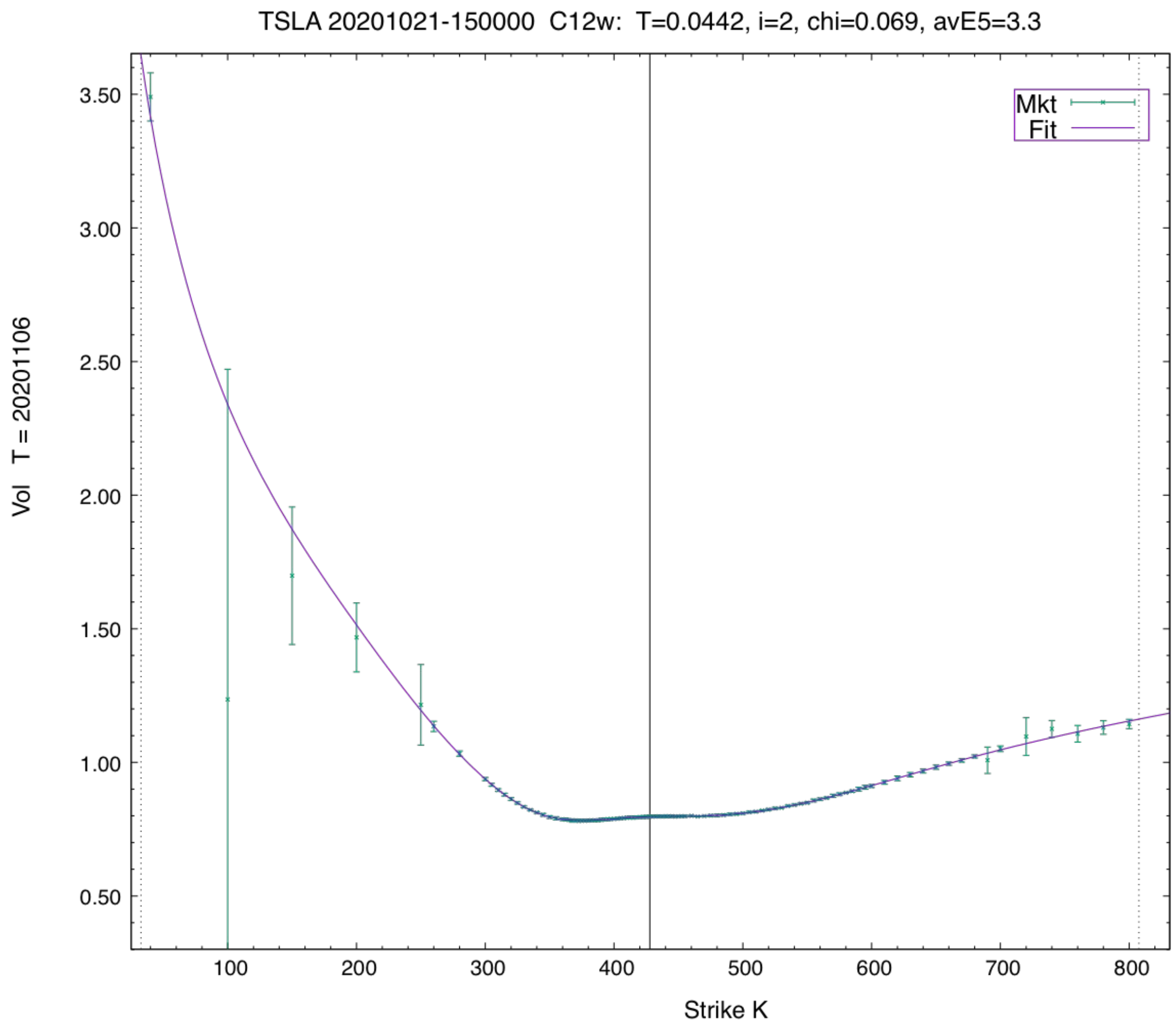
$$z := \mathrm{NS} := \frac{\ln(K/F)}{\sigma_0\sqrt{T}}$$



TSLA 20200403

Do not trade off mids…

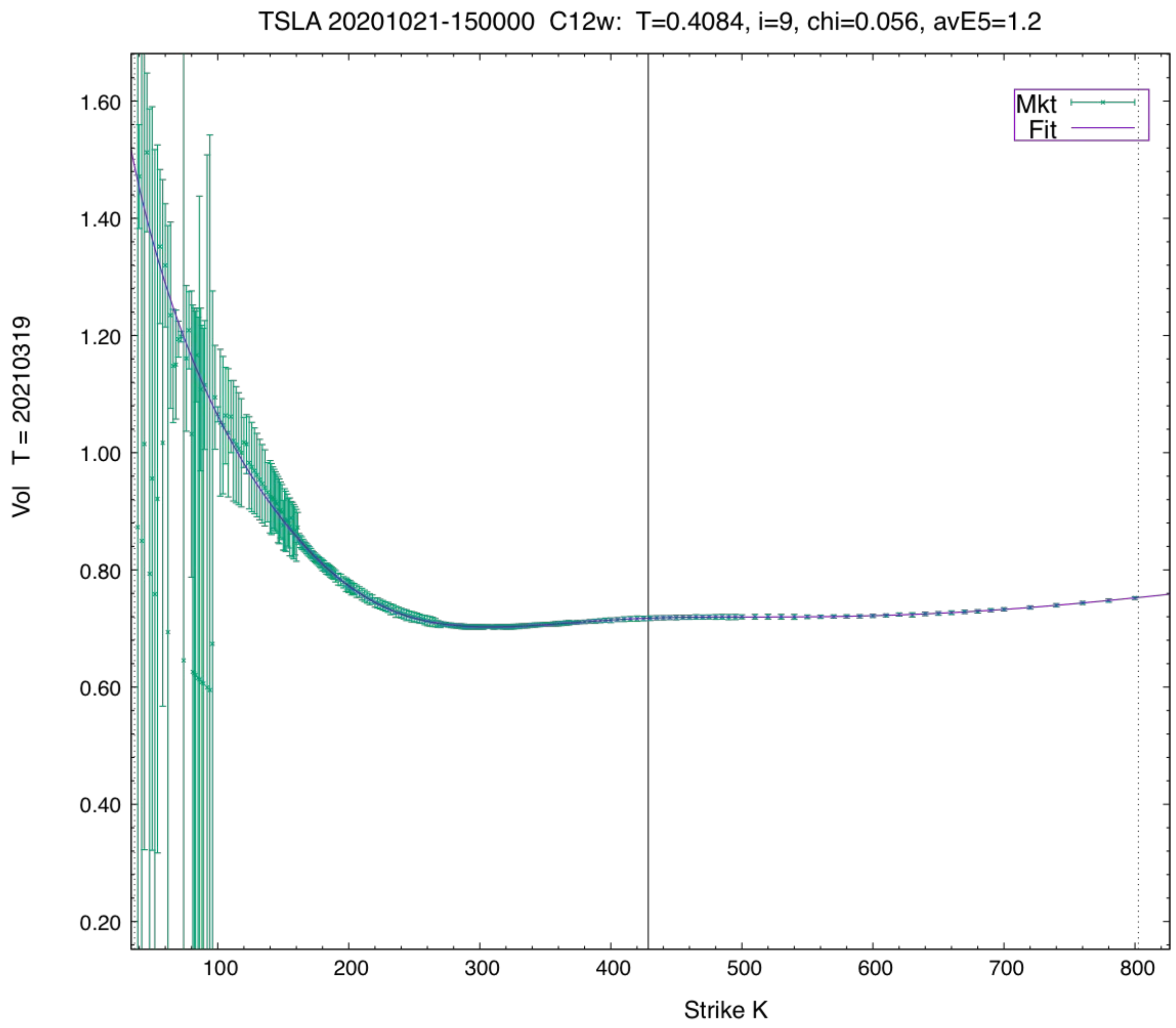
NOTE: Strike range > 10x



TSLA 20201021

Different day -- very different shapes and spreads…

NOTE: Strike range > 10x



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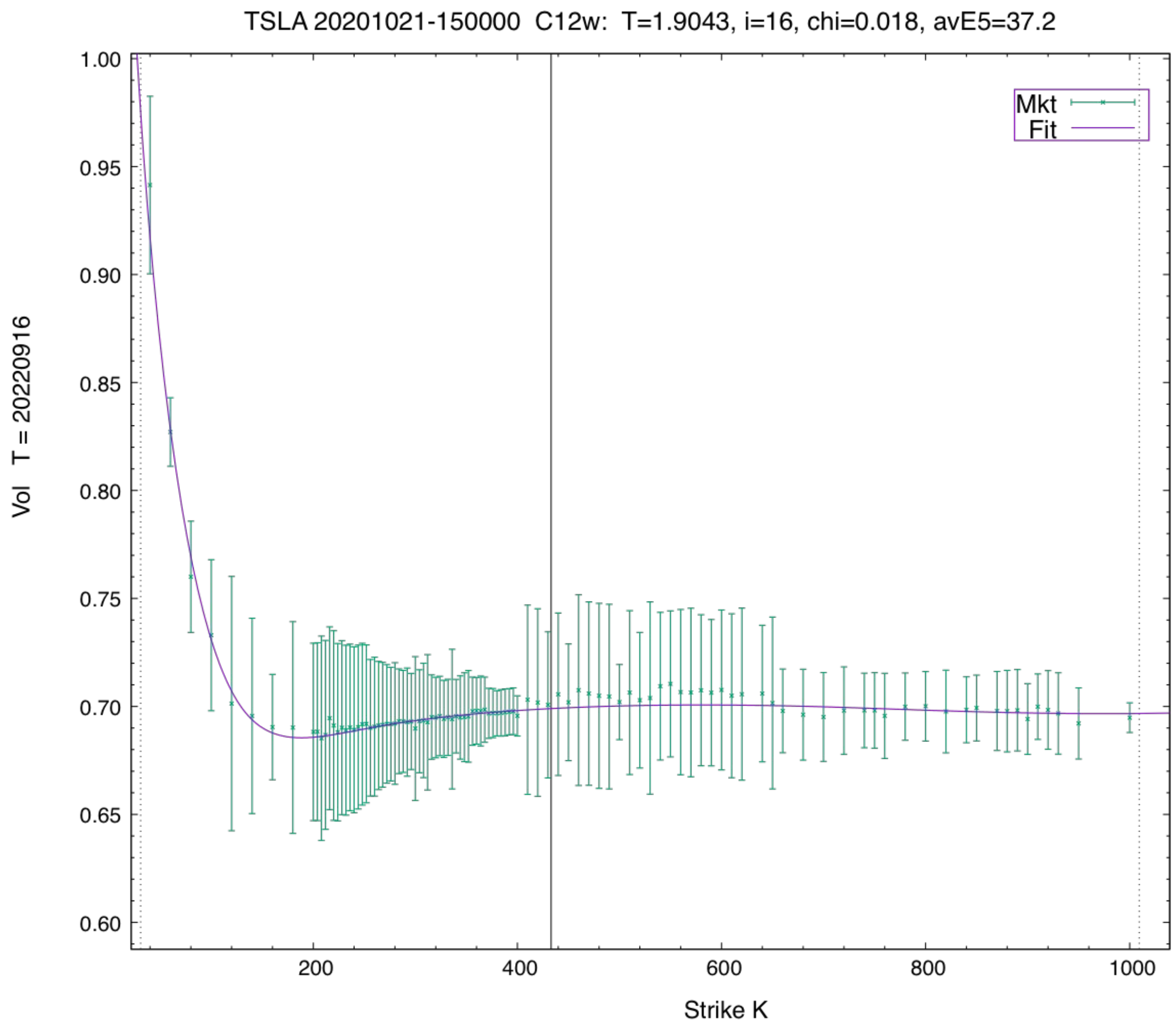
TSLA 20201021

Is the market using the Merton model ?

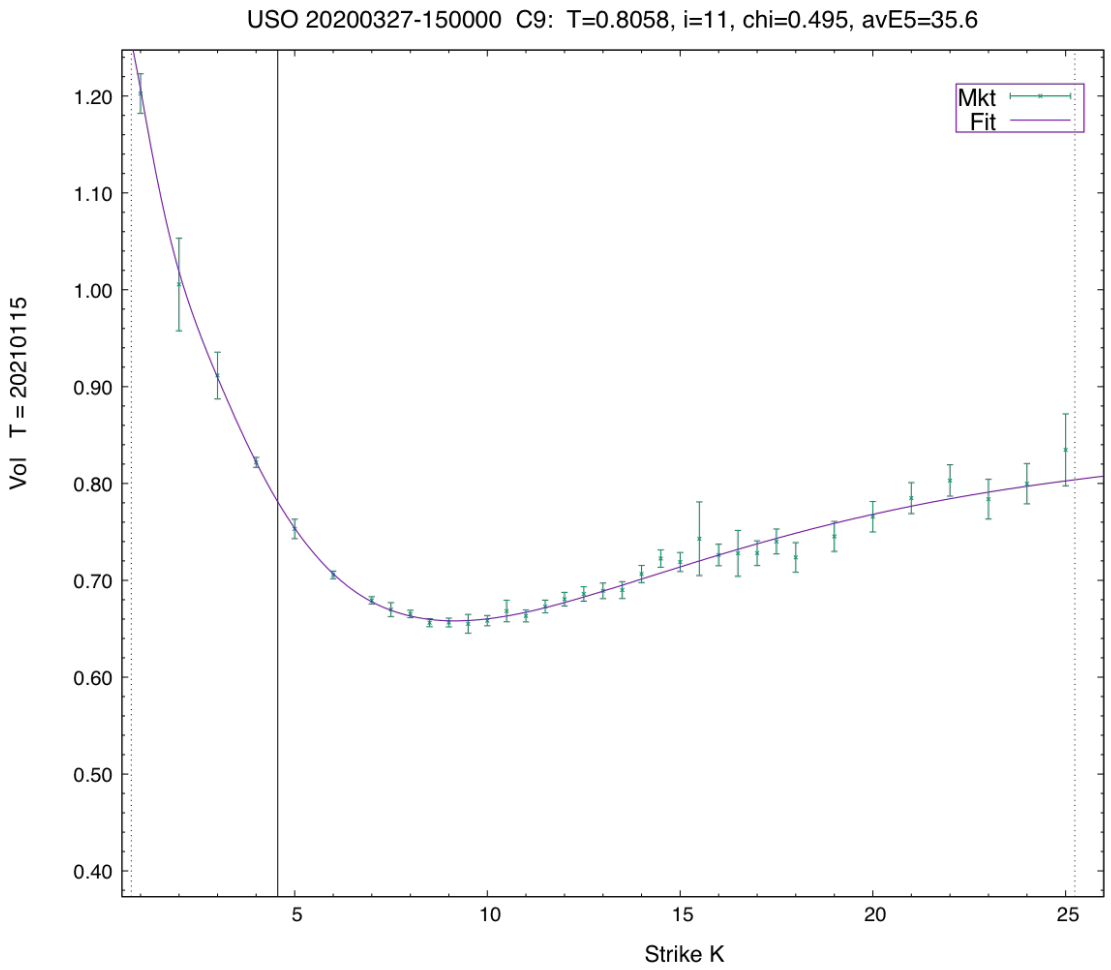
NOTE: Strike range > 10x



TSLA 20201021

Is the market using the Merton model ?

NOTE: Strike range > 10x

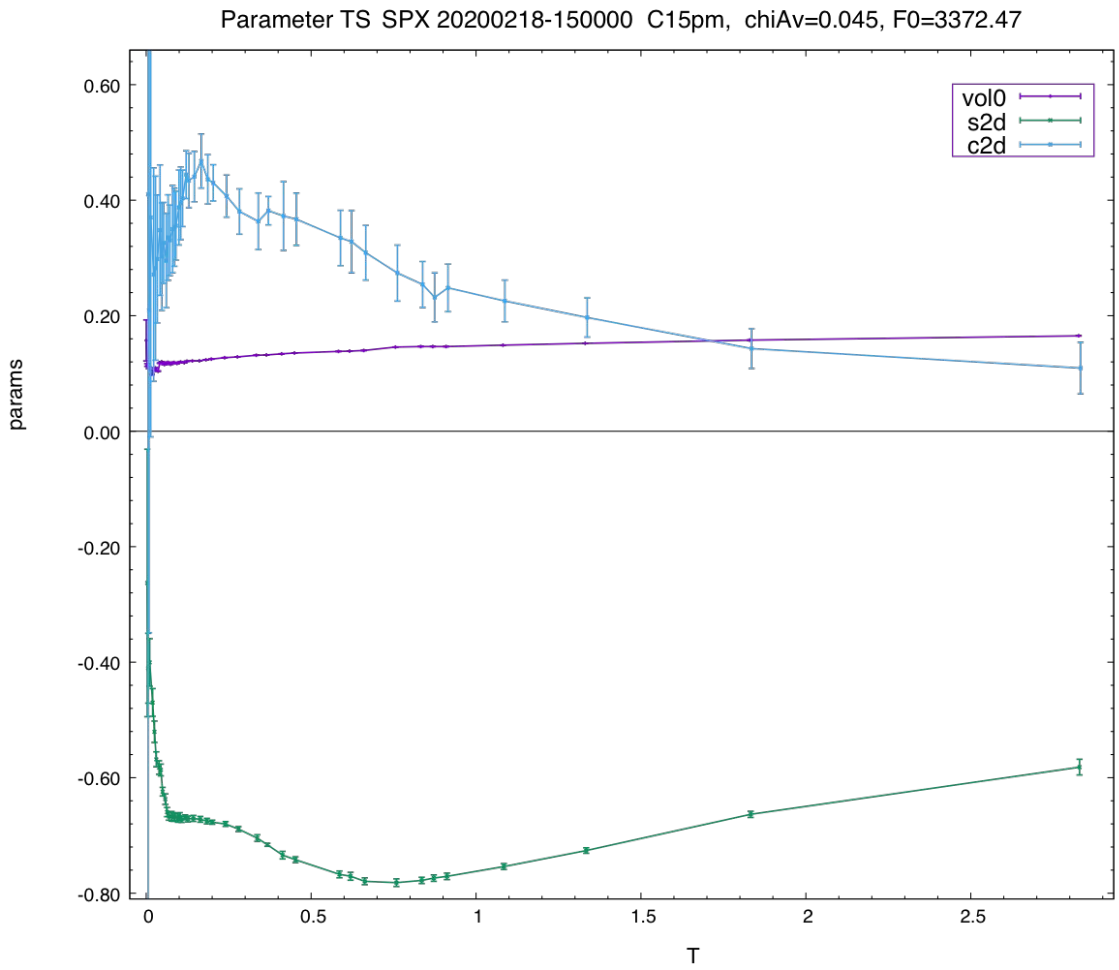


USO 20200327

Do not trade off mids…

5 strikes in a row at $0.03 x 0.04

NOTE: Strike range is 25x



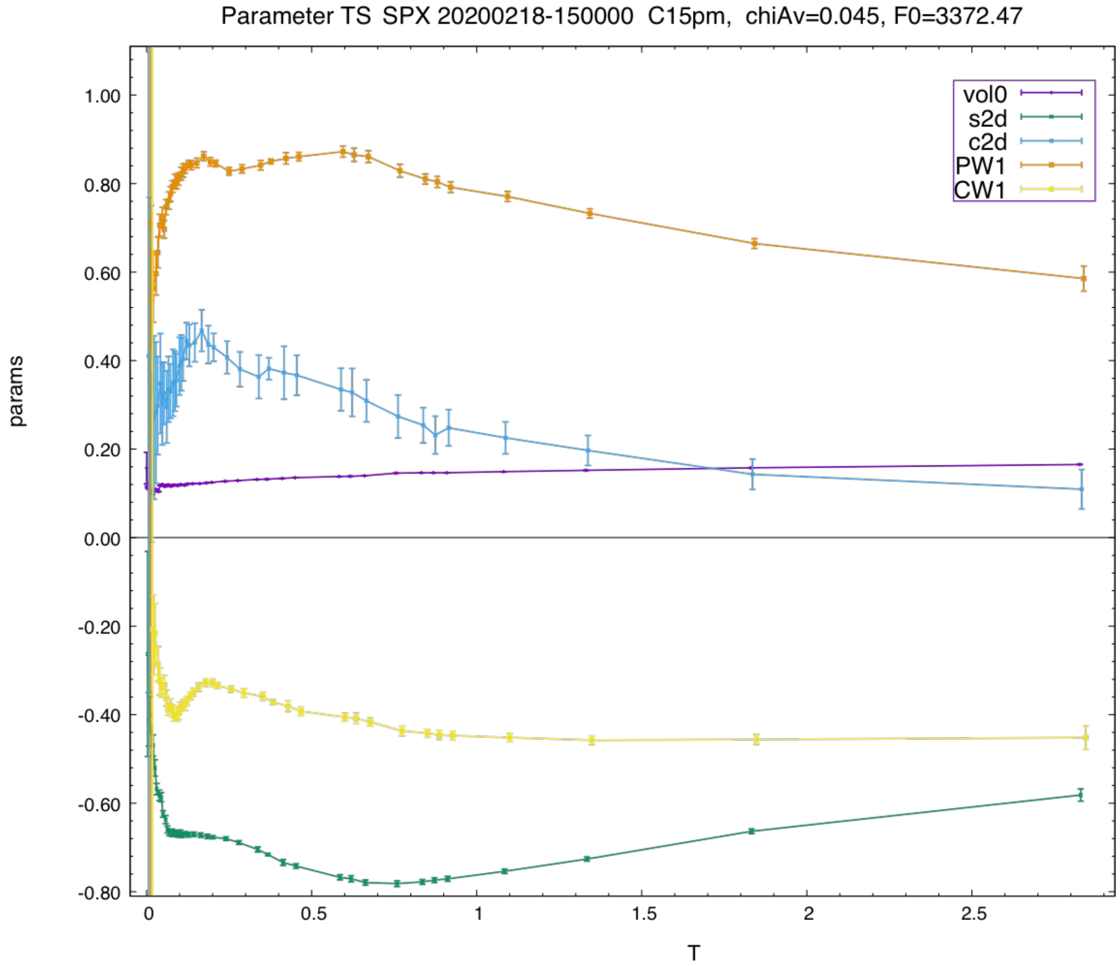
Getting close to March 2020…

SPX 20200218 15:00

C15PM Param Term-Structure

First 3 params…

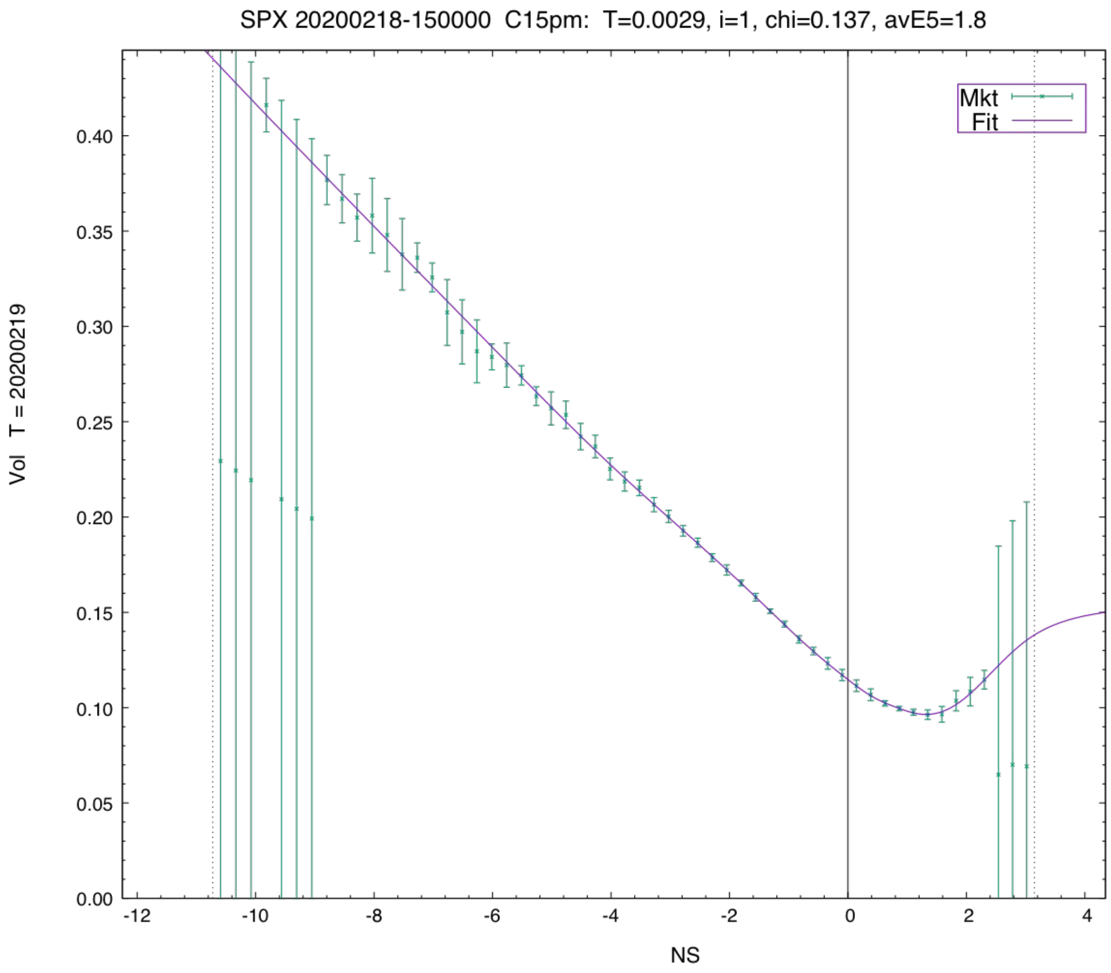
s2(T) a bit unusual…



SPX 20200218 15:00

C15PM Param Term-Structure

First 5 params... meaning?



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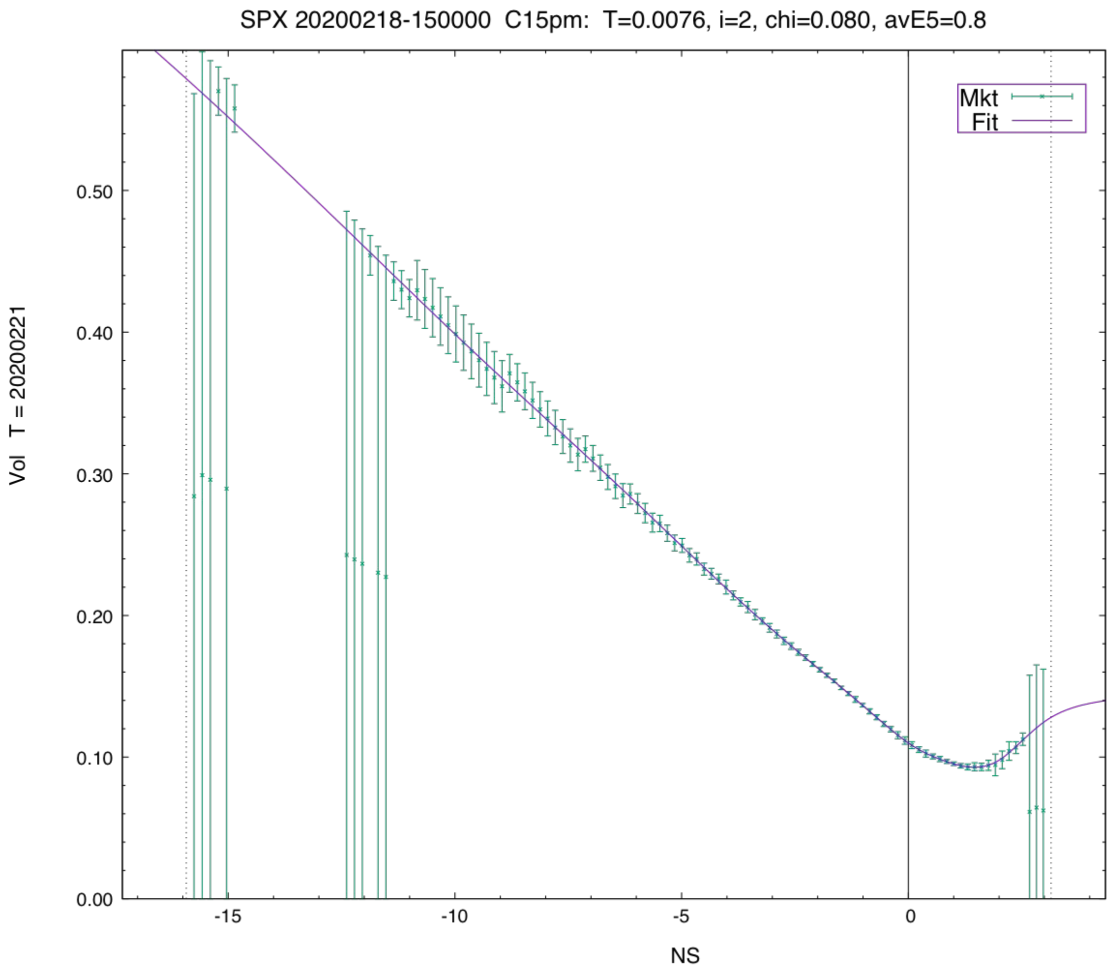
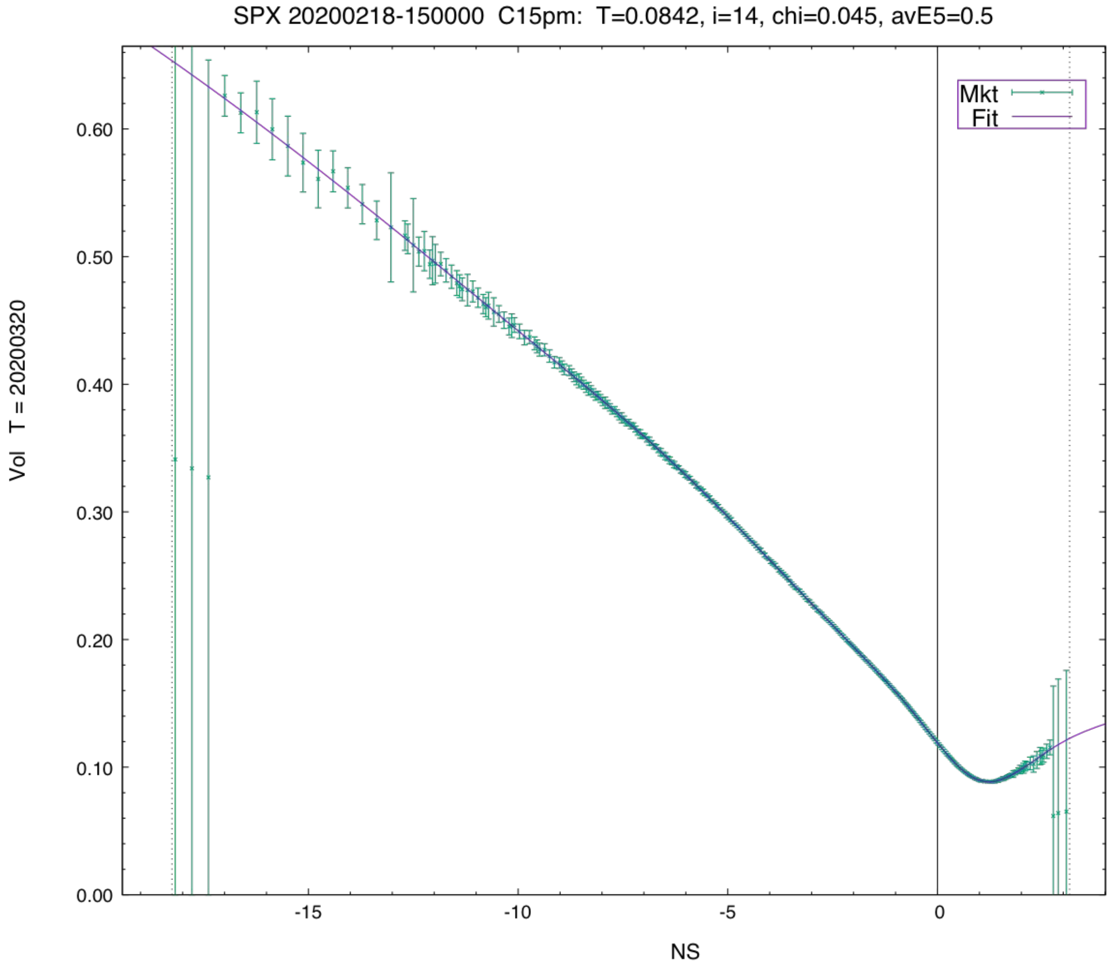
SPX 20200218 15:00

C15PM T = 1d



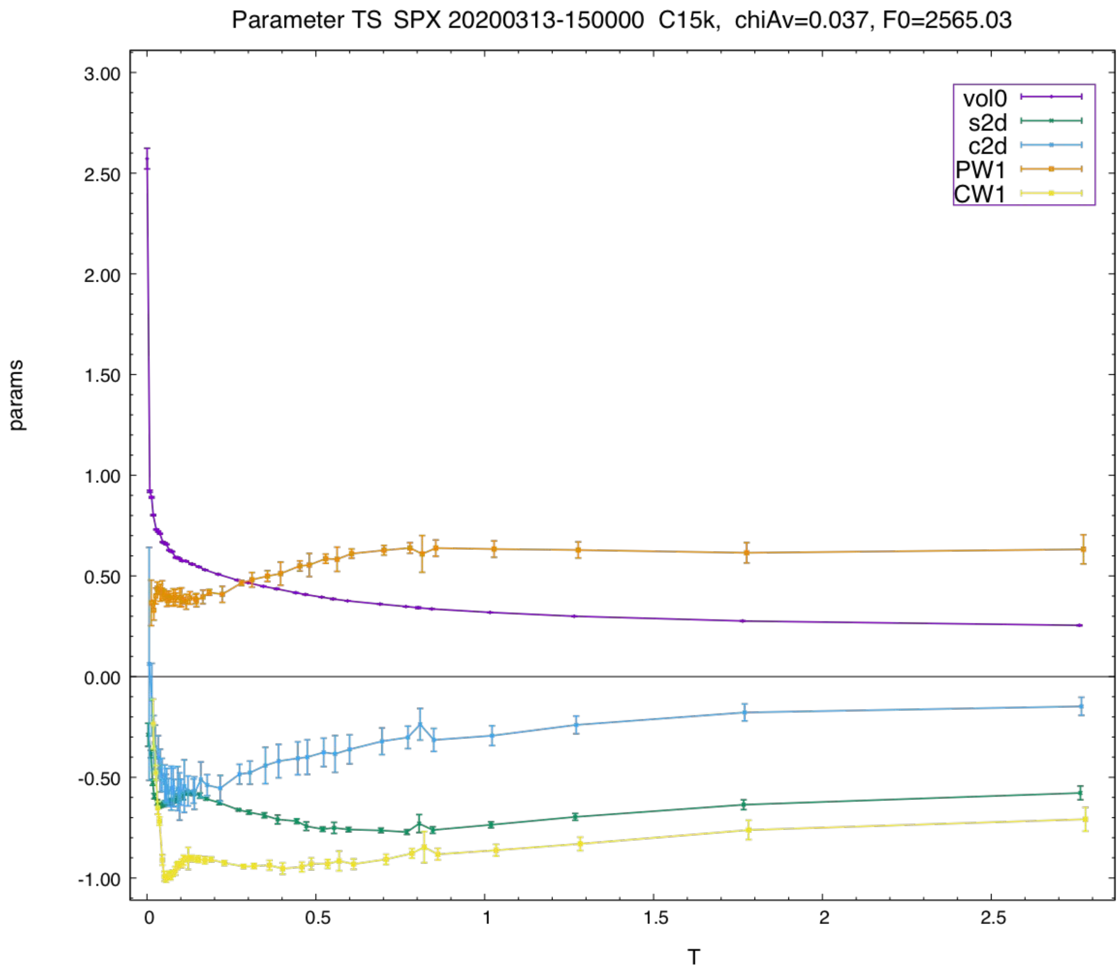
SPX 20200218 15:00

C15PM T = 3d



SPX 20200218 15:00

C15PM T = 1m



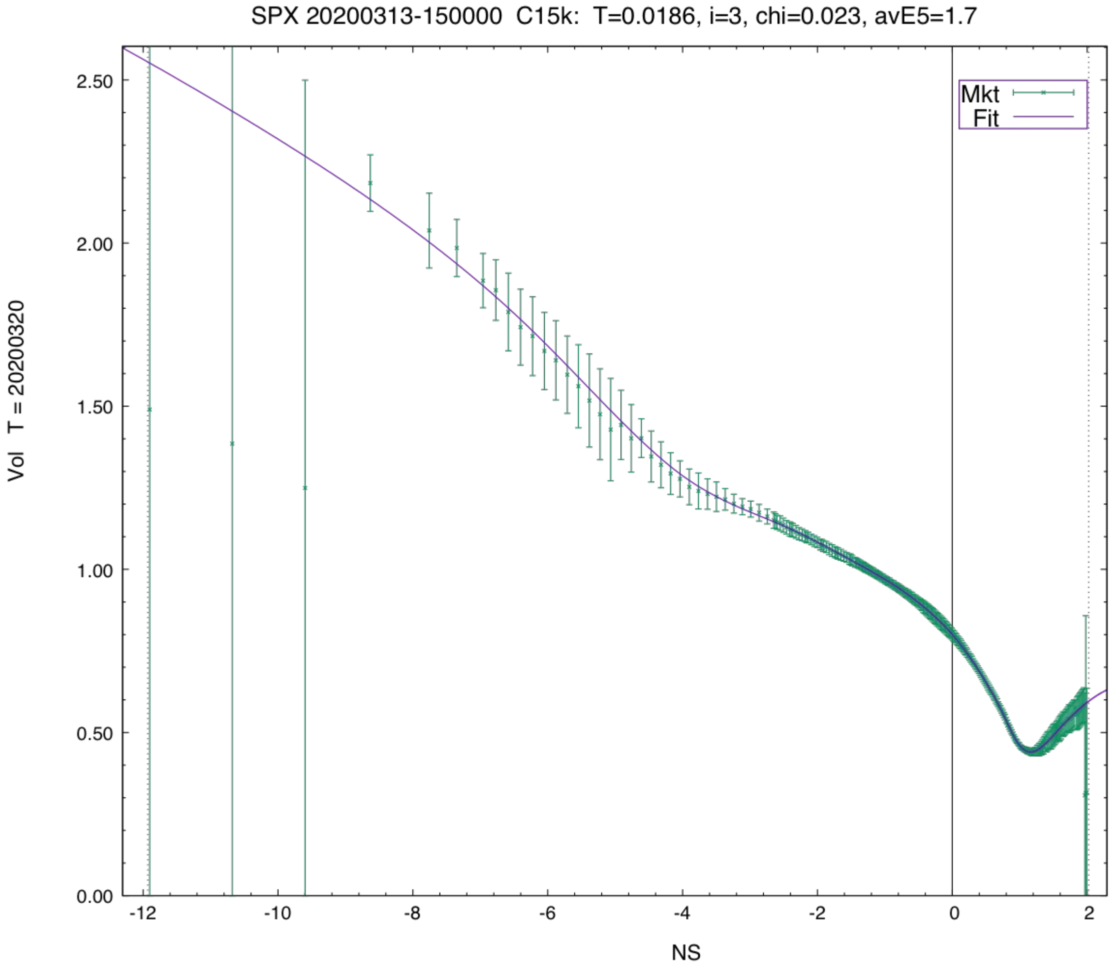
SPX 20200313 15:00

C15K Param Term-Structure during the **covid crash**

First 5 params…

All **c2** < 0 !!

Super-steep near call wing: **CW1**

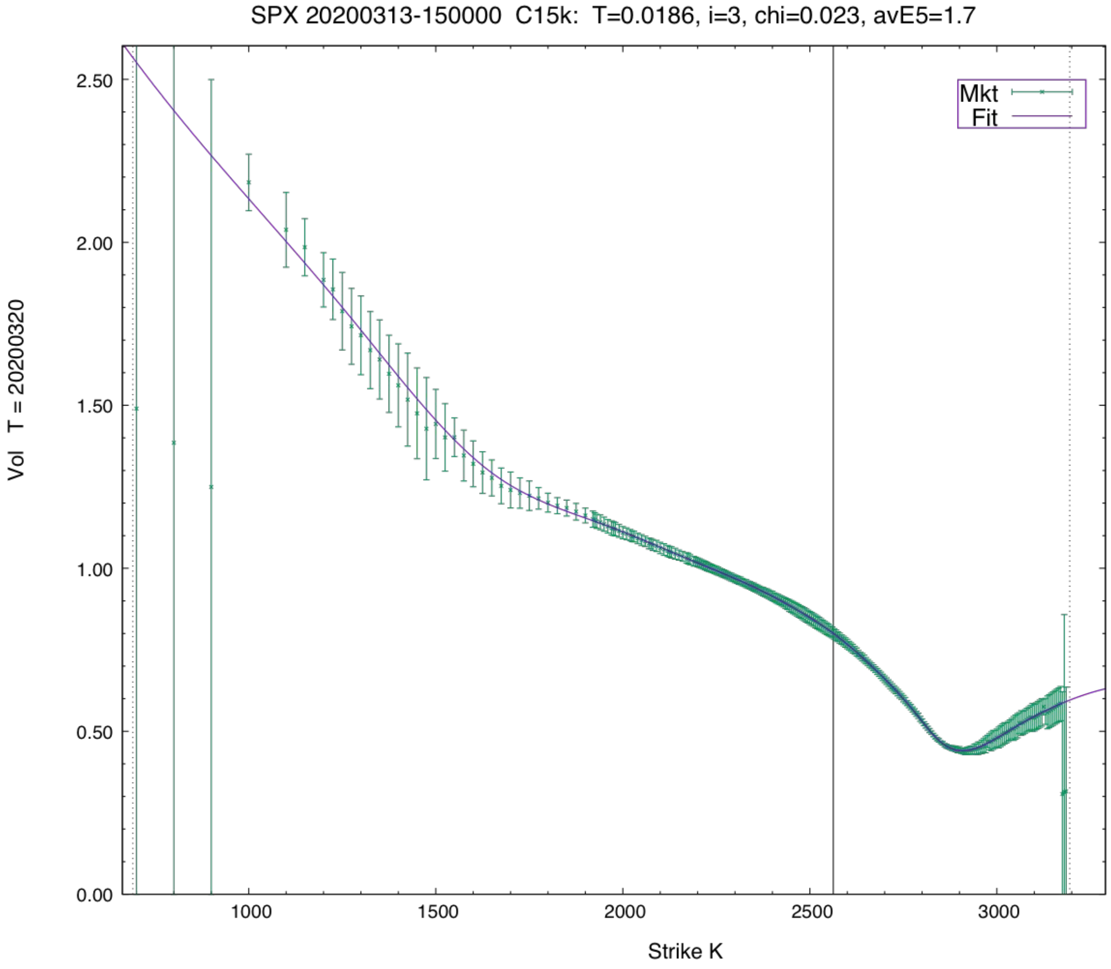


SPX 20200313 15:00

C15K T = 1w, in NS-space

Very compressed CW.

If fit followed PW more closely there would be fly arb…



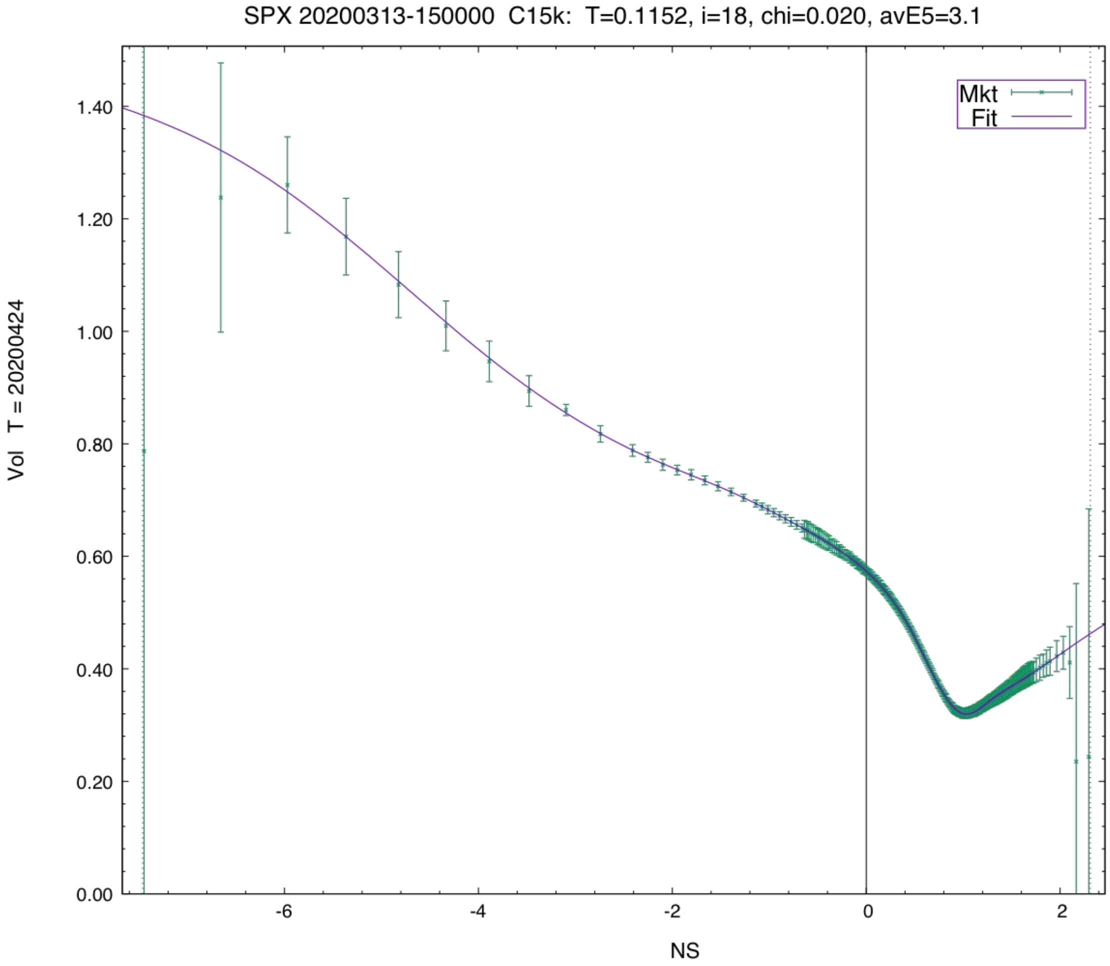
SPX 20200313 15:00

C15K T = 1w, in K-space

Very compressed CW.

If fit followed PW more closely there would be fly arb...

(Pretty well-functioning market over nK=379 strikes here...)



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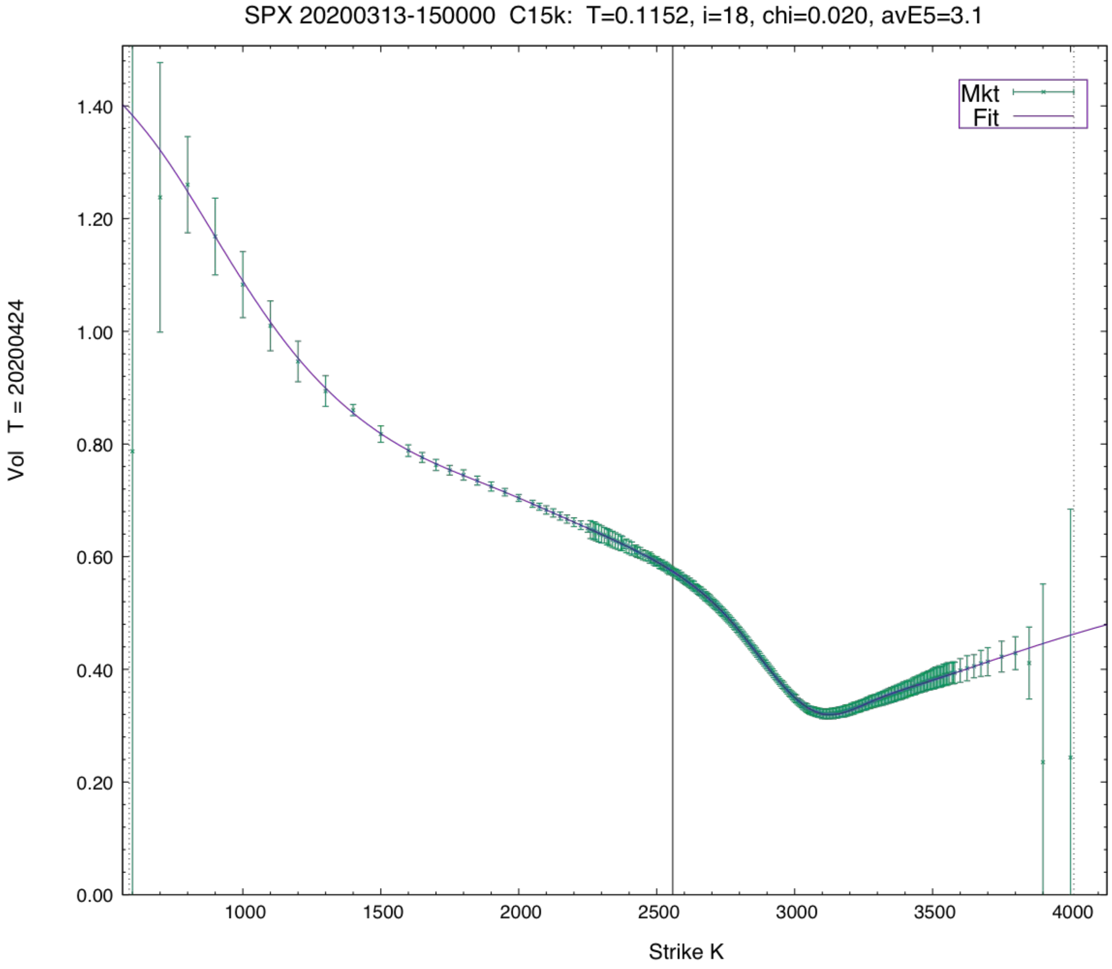
SPX 20200313 15:00

C15K T = 6w, in NS-space

Very compressed CW, very sharp knee...



VOLA DYNAMICS
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SPX 20200313 15:00

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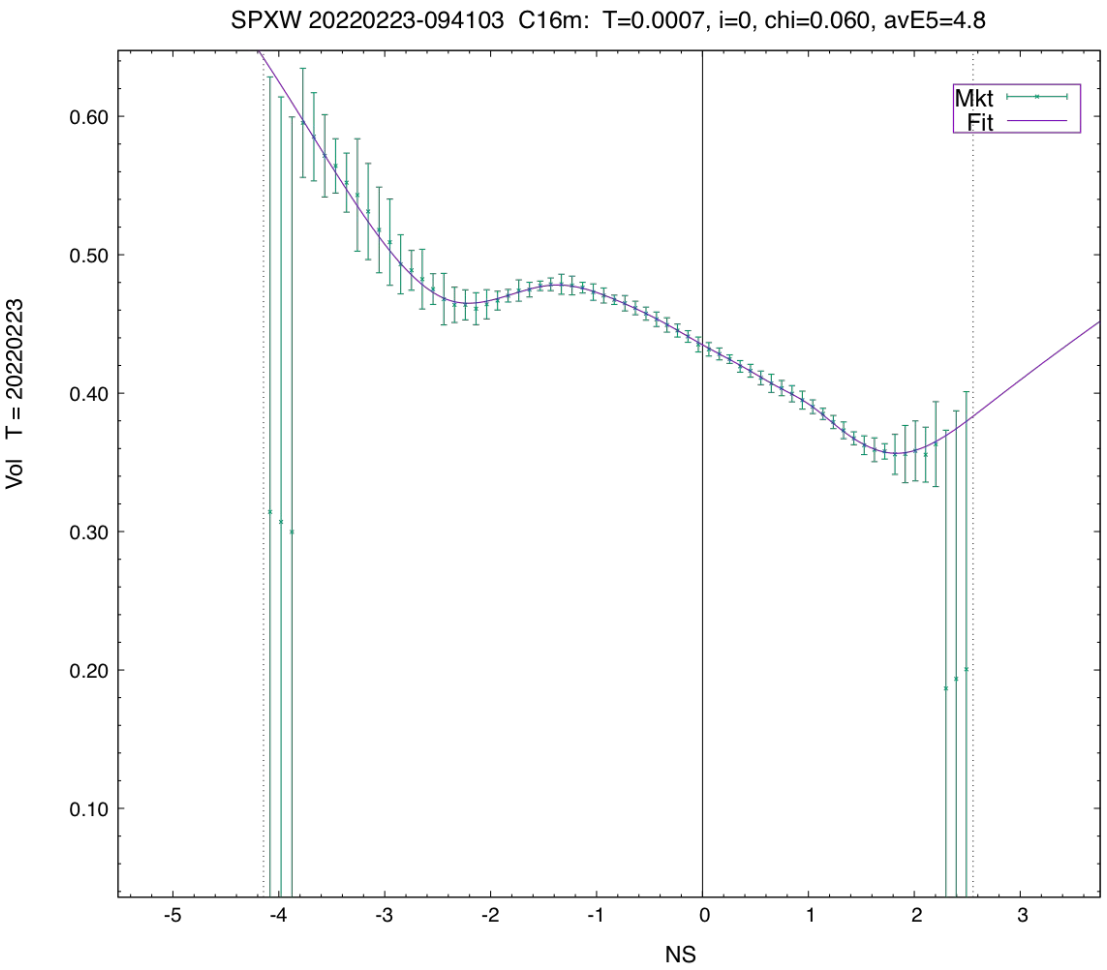


VOLA DYNAMICS
Analytics for Options Trading

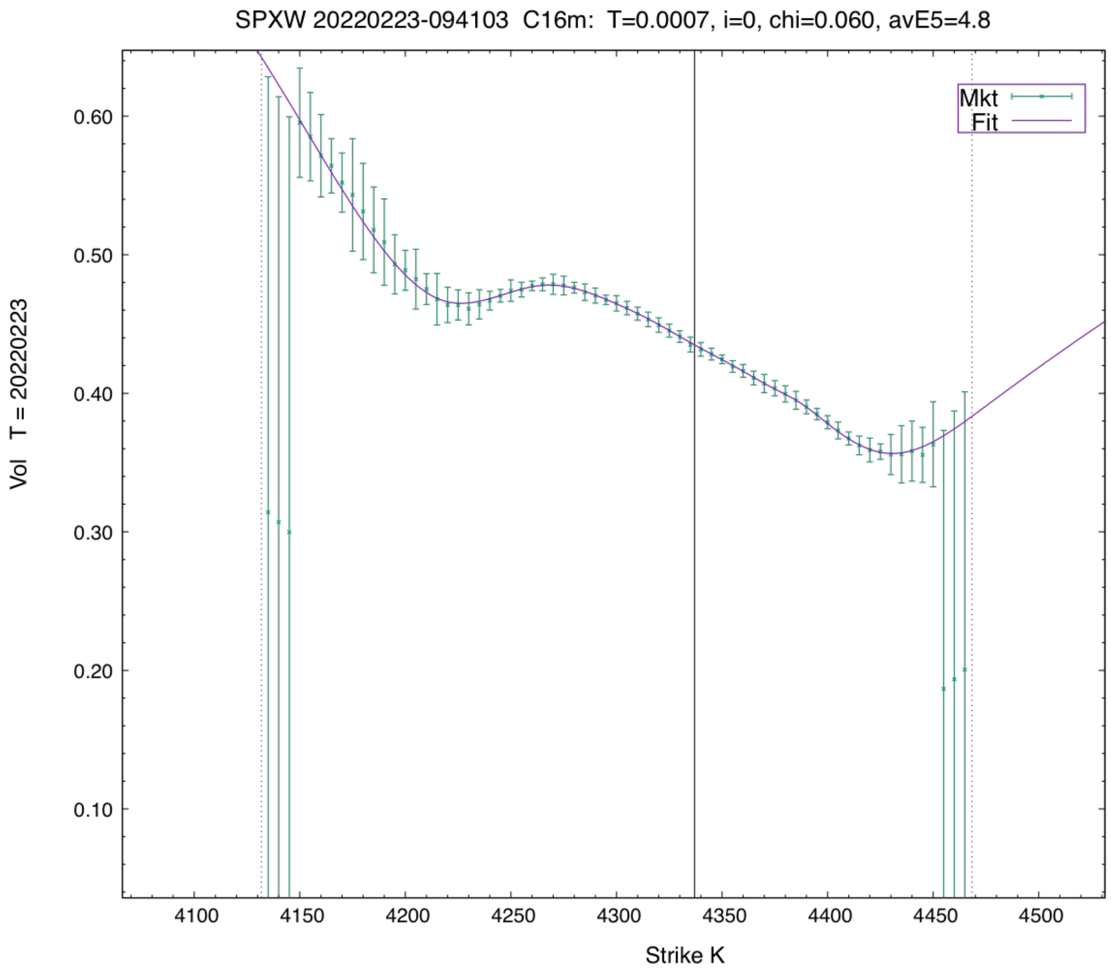
SPX 20220223 9:41:03

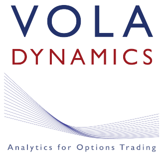
C16m T < 1d, in NS-space

Putin's put wing – shape never seen before!
Pricing a bad & worse scenario?

C16m allows bias-free fits…

Inputs are MP1 here…





SPX 20220223 9:41:03

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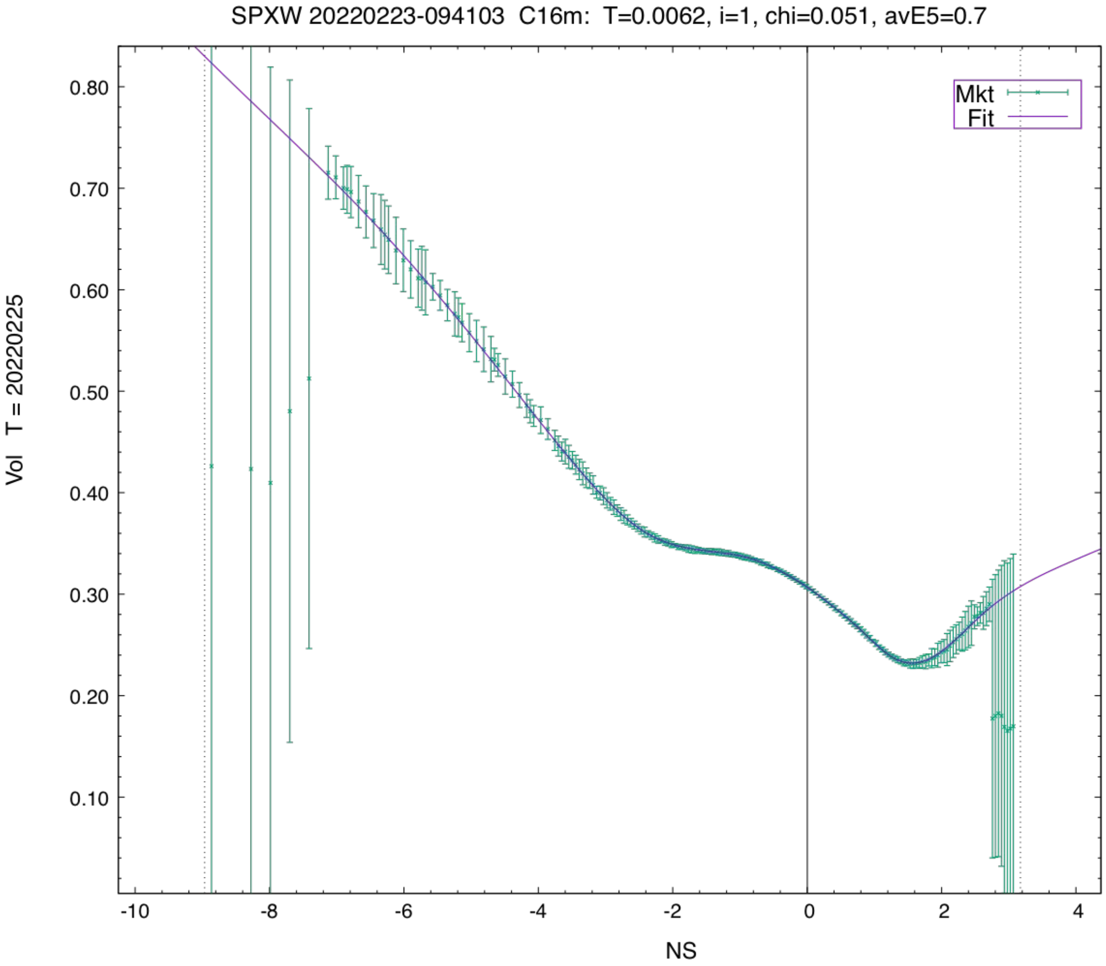

SPX 20220223 9:41:03

C16m T = 2d, in NS-space

What does it all mean ?

- We will explain…
- To do so, let’s take a step back and discuss in more detail:
    - Dividend modeling (briefly)
    - Vol curve/surface parametrizations
    - Arbitrage

Dividend Modeling

- **Three types of dividends**
  - **Yield/borrow**
  - **Cash**
  - **Discrete proportional**
  - Blending scheme to transition from cash to discrete proportional is standard.
- **Two main classes of dividend models** for cash component:
  - **Spot model**
  - **Hybrid models:** Observed stock = "Pure stock" + dividend floor
    - Various flavors, specified by dividend floor details.
    - Same, exact forward formula F = F(S,divs,r,q) for all hybrid models.
    - "Pure stock" still follows GBM.
    - Analytical pricing formulas in Euro case, numerical e.g. grid methods in American case.
    - Allows a lot of easy extensions to handle credit, default, exotics, etc.
  - For details see: *Pricing Vanillas Options with Cash Dividends* (SSRN).

Hybrid Models, Notation

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- In a hybrid model the stock follows *shifted GBM*, and the prices of (un-discounted) European vanillas for the pure stock are:

$$\hat{C} \;=\; +F\,N(d_+) - K\,N(d_-) \tag{1}$$

$$\hat{P} \;=\; -F\,N(-d_+) + K\,N(-d_-) \tag{2}$$

- Here $N(x)$ is the normal cdf, log-moneyness $y := \log(K/F)$, and

$$d_\pm \;:=\; \frac{-y}{\hat{\sigma}} \pm \tfrac{1}{2}\hat{\sigma} \quad , \qquad \hat{\sigma} := \sigma\sqrt{T}$$

- $\sigma = \sigma(T, K)$ is the implied volatility of the option.
- Normalized prices $\hat{V}/F$ are function of two dim-less variables: y, $\hat{\sigma}$.
- Actual prices are obtained by shifting the forward $F = F_T$ and strike K by the shift D_T, that depends on the hybrid model.
- For details: *Pricing Vanilla Options with Cash Dividends* (SSRN).

Our parametrization approach

- Work one term at a time, impose smoothness across terms.
- Factor out overall vol level (ATF) as: $\sigma_0 := \sigma(T, K = F)$.
- Define "shape" curve $f(z) = f(z|\mathbf{p})$ as function of normalized strike (NS)[1]

$$z \;:=\; \frac{y}{\hat{\sigma}_0} \;=\; \frac{\log(K/F)}{\sigma_0\sqrt{T}}$$

such that

$$\sigma(z)^2 \;=\; \sigma_0^2\, f(z|\mathbf{p})$$

- There are no standard definitions – we define dimensionless "skew" and "smile/convexity" as slope and curvature of shape curve:

$$f(z) \;=:\; 1 + s_2\, z + \tfrac{1}{2} c_2\, z^2 + \ldots$$

Our parametrization approach (cont'd)

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- s_2 and c_2 tend to have mild term-structure (except maybe as $T \to 0$). They are even comparable across names. Have been range-bound for decades.
- Sometimes it is useful to work with s_1, c_1 defined via

$$\sigma(z) \;=:\; \sigma_0\,(1 + s_1\,z + \tfrac{1}{2}c_1\,z^2 + \ldots)$$

- Trivially: $s_2 = 2s_1$, $c_2 = 2(c_1 + s_1^2)$.
- Note that

$$\sigma(z) \;=\; \sigma_0 + \frac{s_1}{\sqrt{T}}\,log(K/F) + \ldots,$$

  so that an alternative definition of skew

$$\tilde{s}_1 \;:=\; K\frac{\partial\sigma}{\partial K}|_{K=F} \;=\; \frac{s_1}{\sqrt{T}}$$

- No simple relationships between alternative definitions of curvature/convexity/smile.

No-Arbitrage Constraints in Vol-Space

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- **No butterfly arbitrage**: Implied density ρ should be positive:

$$\hat{C}(T,K) \;=\; \int_0^\infty dS_T \; (S_T - K)_+ \; \rho_T(S_0 \to S_T)$$

$$\Rightarrow \quad \partial_K^2 \hat{C}(T,K) \;=\; \rho_T(S_0 \to S)|_{S=K}$$

- **No calendar arbitrage**: Total BS variance $w(y) := T\sigma(y)^2$ has to be increasing in T at any fixed y.

- Necessary (but generally not sufficient) constraint on the asymptotic wing behavior of implied vols (R. Lee, 2004):

$$w(y) \leq 2|y| \qquad \text{as} \quad |y| \to \infty$$

Specific Curves: Parabolas

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- What are simplest possible implied vol curves? Need at least 3 parameters for ATF behavior.
- Vendors often use

$$\sigma(y)^n = \sigma_0^n \left(1 + s\,y + \tfrac{1}{2}c\,y^2\right) \qquad \text{(or in terms of } z\text{)}$$

- Obviously has arbitrage in wings for $n = 1, 2$.
- Slight hope for $n = 4$, but would imply symmetric wings, which is intuitively and empirically wrong.
- Positivity has to be enforced too.
- Must do better...

Specific Curves: S3/SSVI

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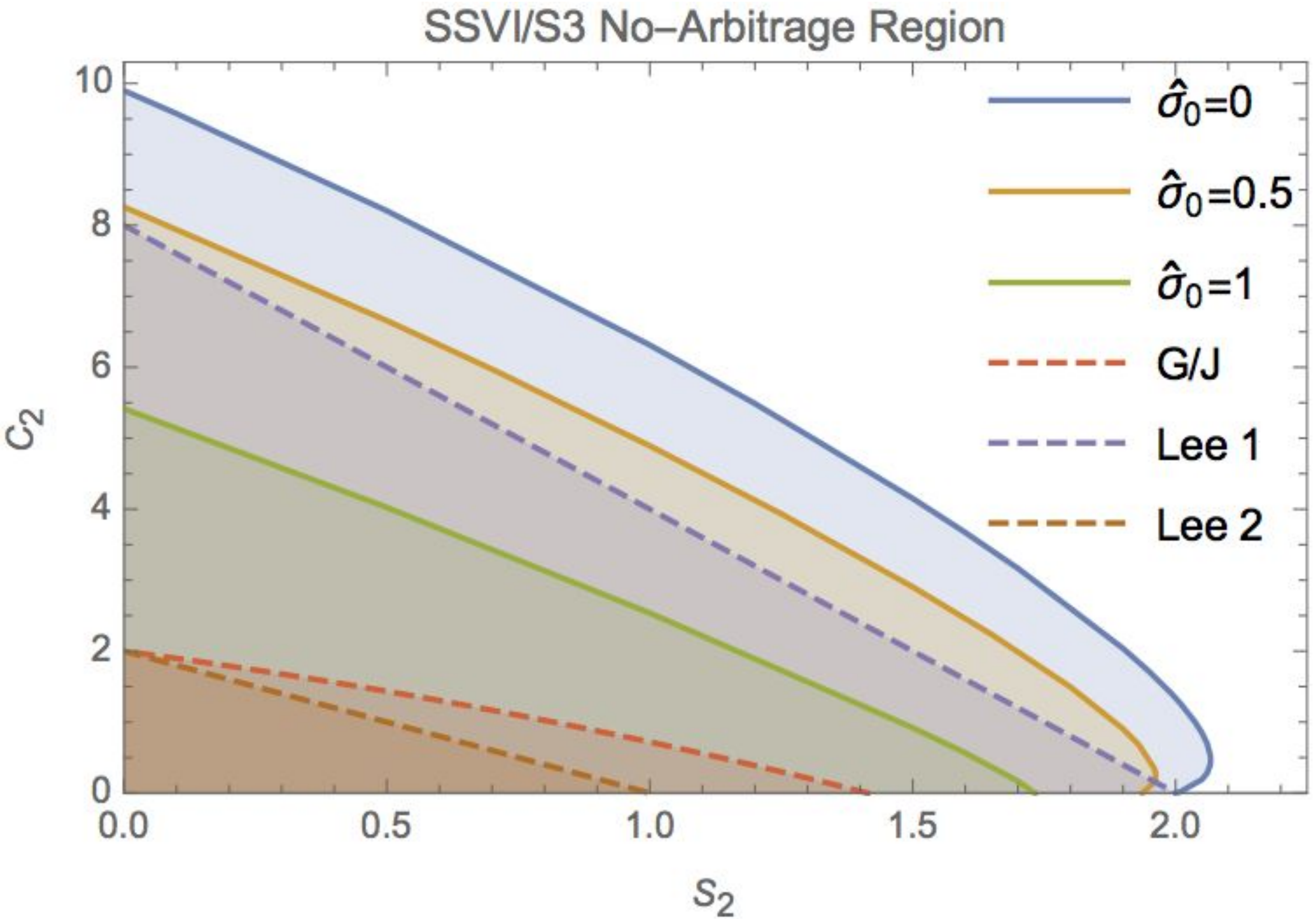
- Simplest sensible curve with 3 parameters ($c_2 \geq 0$):

$$\sigma^2(z) \;=\; \sigma_0^2 \;\left(\tfrac{1}{2}(1+s_2 z) + \sqrt{\tfrac{1}{4}(1+s_2 z)^2 + \tfrac{1}{2}c_2 z^2}\right)$$

- Was independently discovered by TRK (2003, "S3") and Gatheral/Jacquier (2013, "SSVI" = Simple SVI).
- Allows surprisingly varied skew shapes, including "takeover-for-cash" curves as $c_2 \to 0$.
- Allows fitting of vast majority of US equity names.
- Very easy to avoid arbitrage (especially butterfly).
- In fact, in terms of the dimensionless variables $\hat{\sigma}_0, s_2, c_2$ can completely answer the butterfly-arbitrage question...

See our paper on SSRN for details about S3 curve, including simple necessary and sufficient no-butterfly arbitrage conditions in terms of parameters.



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Provable facts about S3 no-fly-arbitrage

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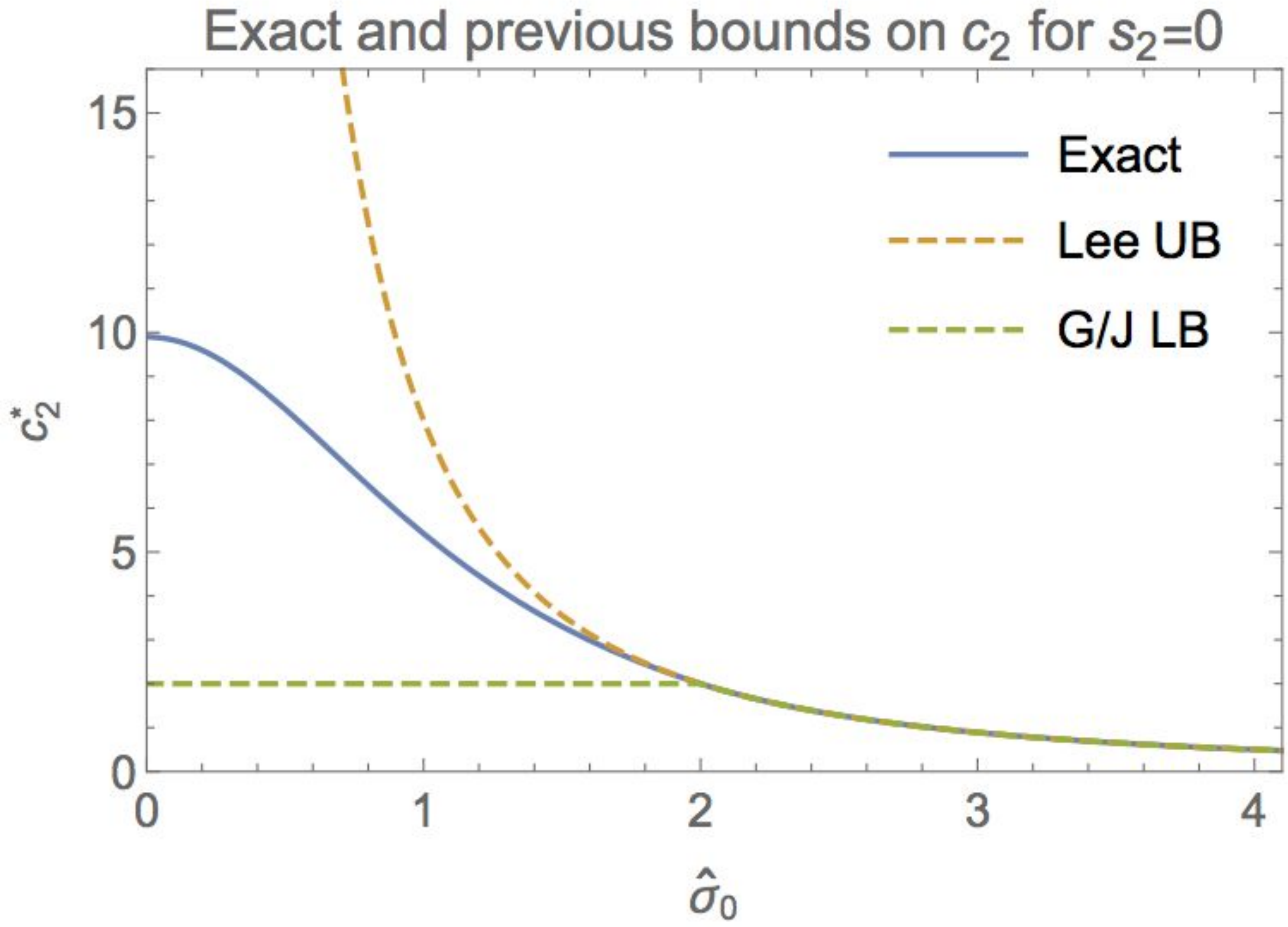
Theorem 1: When $c_2 = 0$, the SSVI/S3 curve has no butterfly arbitrage if and only if $|s_2| \leq s_2^\star(\hat{\sigma}_0)$, where

$$s_2^\star(\hat{\sigma}_0)^2 \;:=\; \begin{cases} 4 - \hat{\sigma}_0^2 & \text{for} \quad \hat{\sigma}_0^2 \leq 2 \\ 4/\hat{\sigma}_0^2 & \text{for} \quad \hat{\sigma}_0^2 \geq 2 \end{cases}$$

Theorem 2: When $s_2 = 0$, the SSVI/S3 curve has no butterfly arbitrage if and only if $c_2 \leq c_2^\star(\hat{\sigma}_0)$, where

$$c_2^\star(\hat{\sigma}_0) \;:=\; \begin{cases} \dfrac{5-\frac{1}{8}\hat{\sigma}_0^2}{\left(1-\frac{1}{8}\hat{\sigma}_0^2\right)^2+\hat{\sigma}_0^2} + \sqrt{\left(\dfrac{5-\frac{1}{8}\hat{\sigma}_0^2}{\left(1-\frac{1}{8}\hat{\sigma}_0^2\right)^2+\hat{\sigma}_0^2}\right)^2 - \dfrac{1}{\left(1-\frac{1}{8}\hat{\sigma}_0^2\right)^2+\hat{\sigma}_0^2}} & \text{for} \quad \hat{\sigma}_0^2 \leq 4 \\ 8/\hat{\sigma}_0^2 & \text{for} \quad \hat{\sigma}_0^2 \geq 4 \end{cases}$$



Theorem 2
visualized

More about no-arbitrage…

In terms European un-discounted call prices $\hat{C}(T, K)$ we are all familiar with:

Definition (No-Arbitrage): A call price surface, $\hat{C}(T, K)$, defined for all $T \geq 0, K \geq 0$ (or some subset thereof) is free of static arbitrage if:

1. $\hat{C}(T, K)$ is continuous and non-increasing in K.
2. $\hat{C}(T, K)$ is convex in K.
3. $\hat{C}(T, K \to \infty) = 0$.
4. In terms of a *forward curve*, $F_T > 0$: $(F_T - K)_+ \leq \hat{C}(T, K) \leq F_T$, $\hat{C}(0, K) = (F_0 - K)_+$.
5. $\hat{C}(T, K)/F_T$ is non-decreasing in T at fixed K/F_T.

Remark 1: No differentiability in K required, it follows from convexity for all except a discrete set of K!

Remark 2: Condition 4 holds automatically if prices are parametrized in terms of Black formula. PCP too.

No-Arbitrage in Vol Space:

Translating the price-space no-arbitrage conditions into vol-space, we get:

Definition (No-Arbitrage in Volatility Space): A normalized implied volatility surface, $\hat{\sigma}(T, y)$, used to parametrize prices via the Black formula for calls and puts (hence PCP holds) is free of static arbitrage if:

1. $\hat{\sigma}(T, y) > 0$ for all y (and $T > 0$) is continuous in y.
2. $\hat{\sigma}(T, y)$ is twice differentiable in y, except perhaps for a discrete set of y.
3. The density factor, $g(y)$, is non-negative, $g(y) \geq 0$.
4. $d_+ \to -\infty$ as $y \to +\infty$, for any $T > 0$.
5. $\hat{\sigma}(T, y)$ is non-decreasing in T at fixed y.

The density factor

The implied density can be written as

$$\rho_y(y) = \frac{g(y)}{\hat{\sigma}(y)} n(d_-(y))$$

in terms of the density factor (aka “g-function”) appearing in the no-butterfly-arb condition 3:

$$g(y) = \left(1 - \frac{y\partial_y w(y)}{2w(y)}\right)^2 - \frac{1}{4}\left(\frac{1}{w(y)} + \frac{1}{4}\right)\partial_y w(y)^2 + \frac{1}{2}\partial_y^2 w(y)$$

- In a Black-Scholes universe: $g(y) = 1$.
- Too much negative curvature (last term) can lead to $g(y) < 0$.
- There are different ways of writing g(y). Analyzing $g(y) > 0$ for some non-trivial curve parametrization always gets hard quickly! (S3 is by far the easiest, but not trivial...)

ATF No-Arbitrage Constraints

- If $w(z) = \hat{\sigma}_0^2\,(1 + s_2 z + \frac{1}{2} c_2 z^2 + \ldots)$, then

$$g(z{=}0) = 1 + \tfrac{1}{2} c_2 - \tfrac{1}{4} s_2^2\,(1 + \tfrac{1}{4}\hat{\sigma}_0^2)$$

- $g(0) \geq 0$ implies upper bound on slope

$$s_2^2 \leq \frac{4 + 2c_2}{1 + \frac{1}{4}\hat{\sigma}_0^2}$$

or lower bound on curvature ($c_1 = \frac{1}{2} c_2 - \frac{1}{4} s_2^2$)

$$c_1 \geq -1 + \tfrac{1}{16}\, s_2^2\, \hat{\sigma}_0^2 \approx -1$$

- Very relevant around FOMC and earnings where not just $c_1 < 0$ but even $c_2 < 0$ can happen!

Similarly, with a slightly larger correction term:

$c_2 \gtrsim -2$

More fun with arbitrage

- We all know the "**global**" no-strike-arbitrage condition:
  - $\rho(y) \geqslant 0$ for all y $\iff$ No strike arbitrage for any y in this term.
  - In words: No butterfly arbitrage ⇔ No strike arbitrage of any sort.
- What is the "**local**" no arb condition, for a given y?
  - $\rho(y) \geqslant 0$ excludes butterfly-arbitrage in y.
  - But there can still be (strike-)spread arbitrage in y!

- Win a Vola hoodie if you can name the necessary and sufficient local condition in the nicest possible manner!

More fun with arbitrage...

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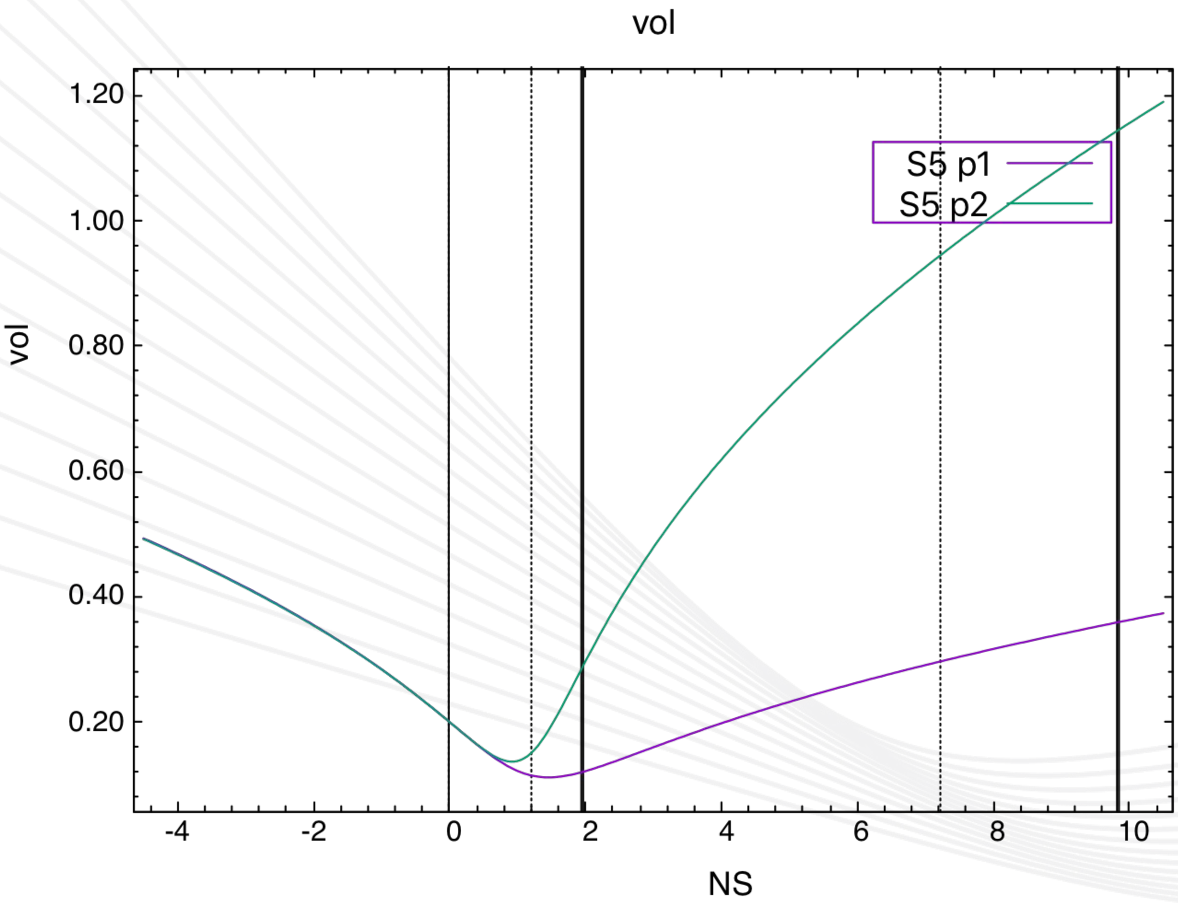
- The necessary and sufficient local no-strike-arbitrage condition in y is:
  - **pdf(y) ⩾ 0** and **0 ⩽ cdf(y) ⩽ 1**
- Proof: The cdf is by definition

$$c(y) := \int_{-\infty}^{y} dy' \, \rho_y(y') = \int_0^{K(y)} dS_T \, \rho_K(S_T)$$

But this is also $\partial_K \hat{P}$ and $\partial_K \hat{C}$ – 1. So:

- Put spread arbitrage, $\partial_K \hat{P} < 0 \iff c(y) < 0$.
- Call spread arbitrage, $\partial_K \hat{C} > 0 \iff c(y) > 1$.
- The cdf c(y) always goes to 1 at large y, even when call prices do not go to 0!
- Spread arb implies fly arb, but not vice versa (Proof: obvious). In fact:
  - **Max spread arb ⩽ max fly arb** !

Examples of spread and fly arbitrage: S5



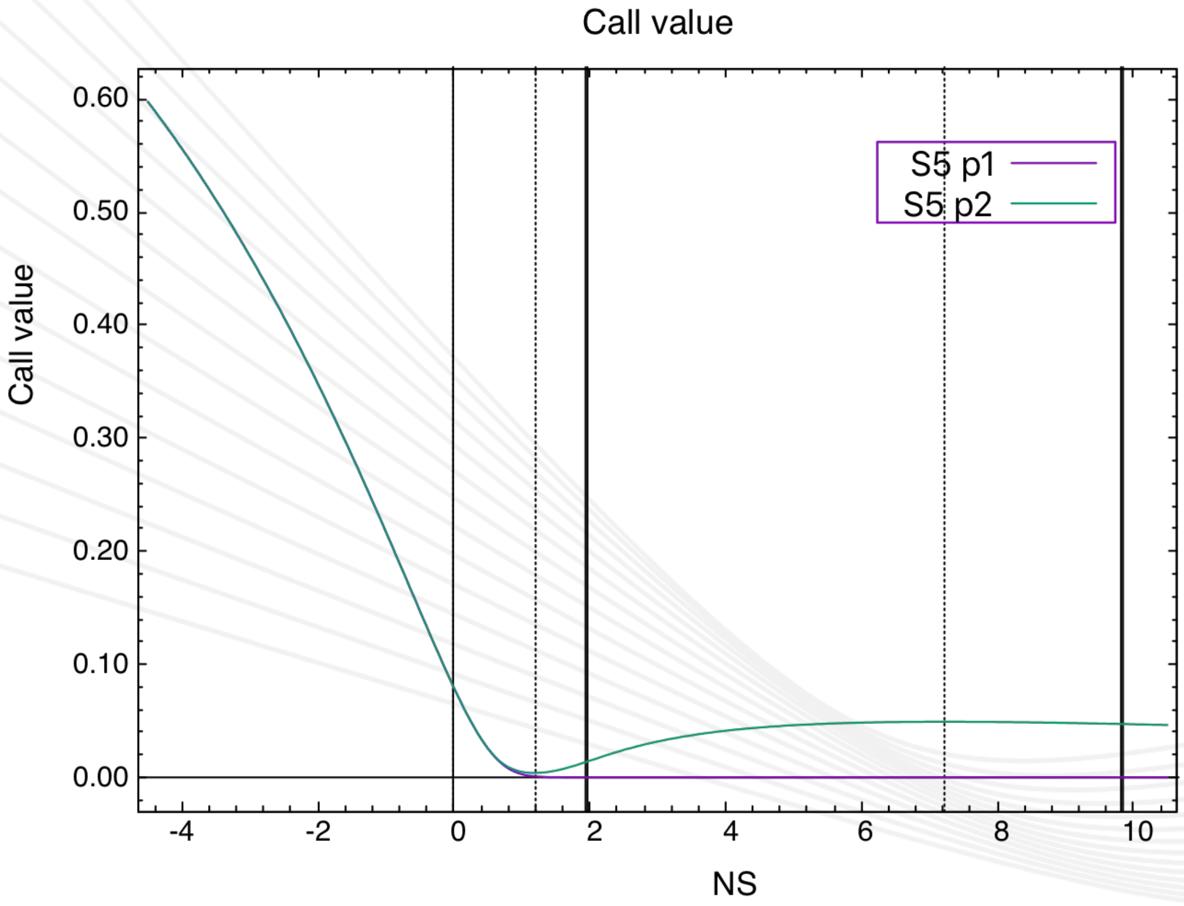
The Lee bounds are **not** violated:

dw/dy=0.79 in p2 far CW

Asymptotically there is no arb…

S5 = SVI

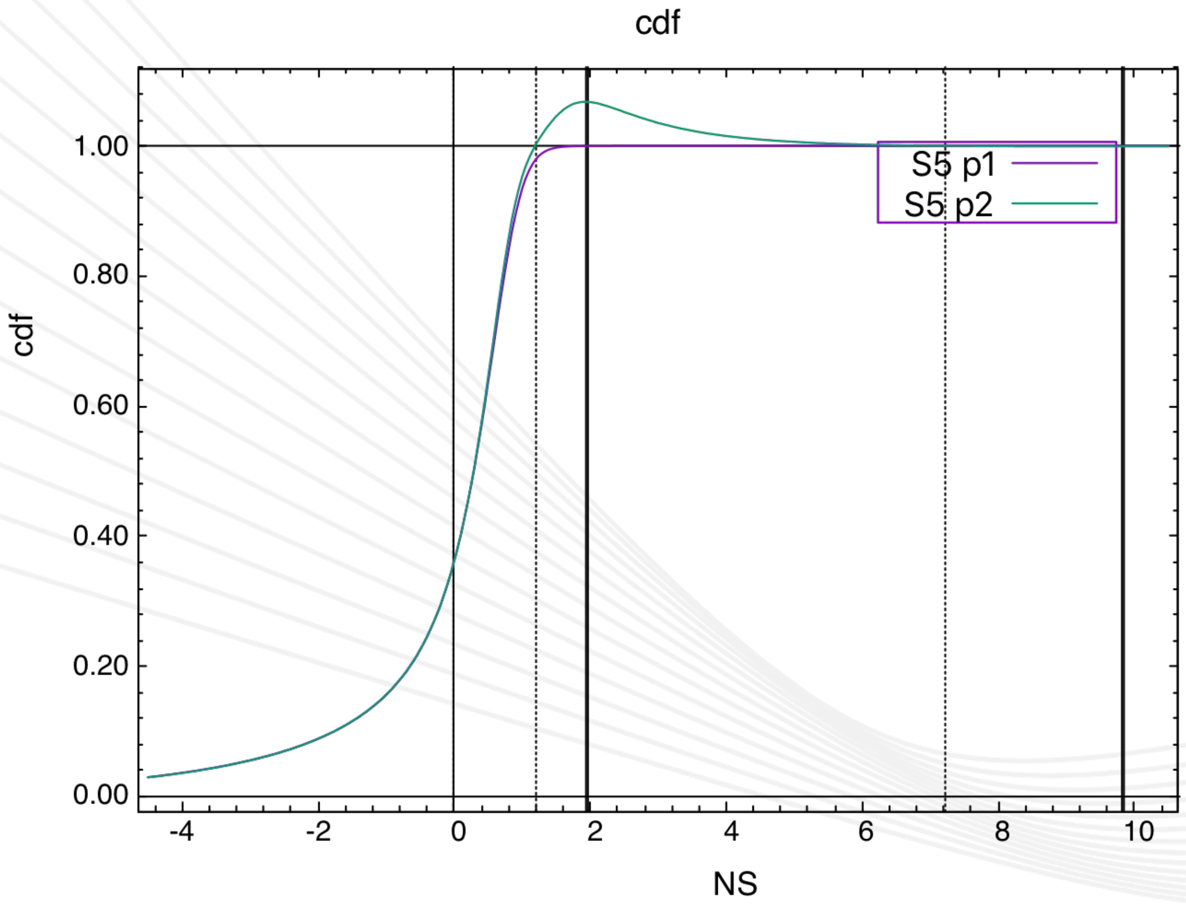
Examples of spread and fly arbitrage: S5



Now we know what the dotted lines mean…

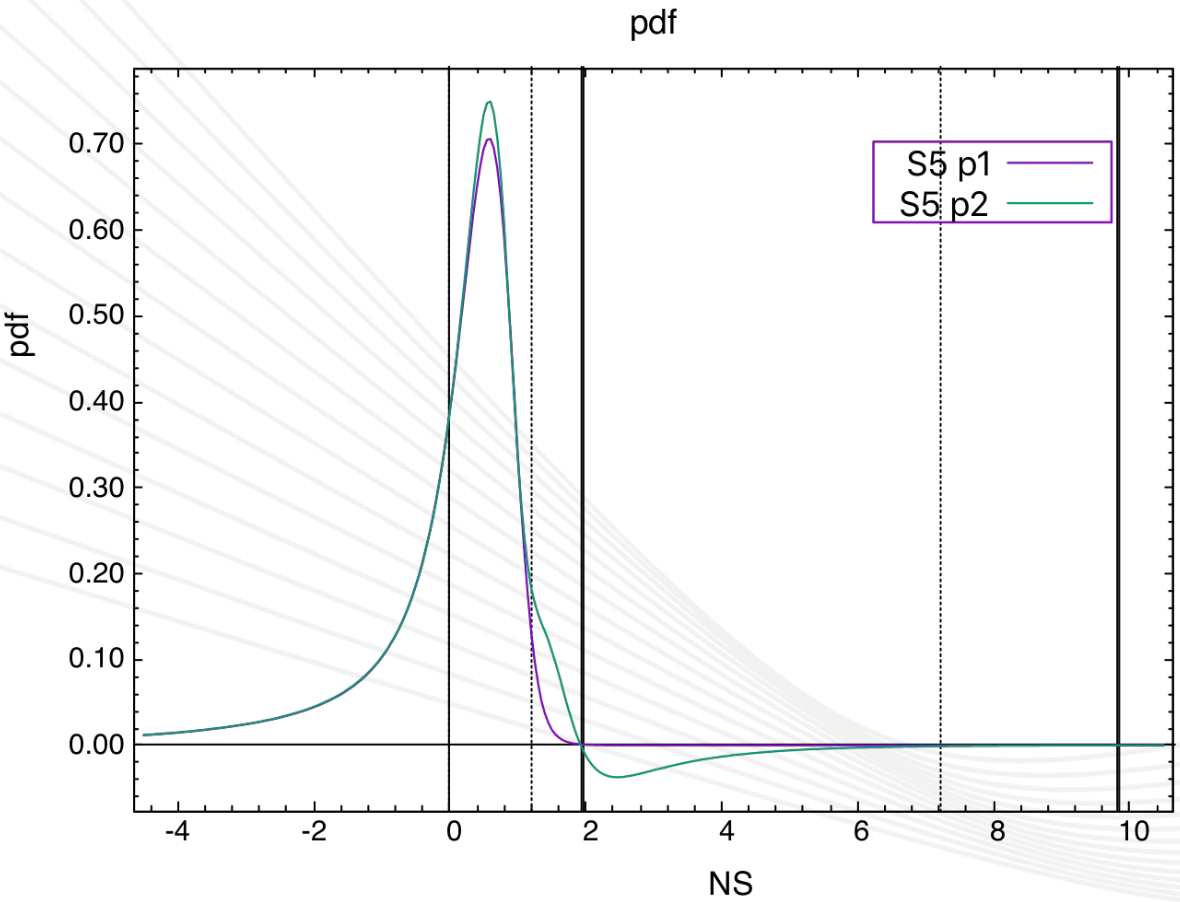
Note: The convexity relevant for fly-arb is for C(K) not C(NS), but…

Examples of spread and fly arbitrage: S5



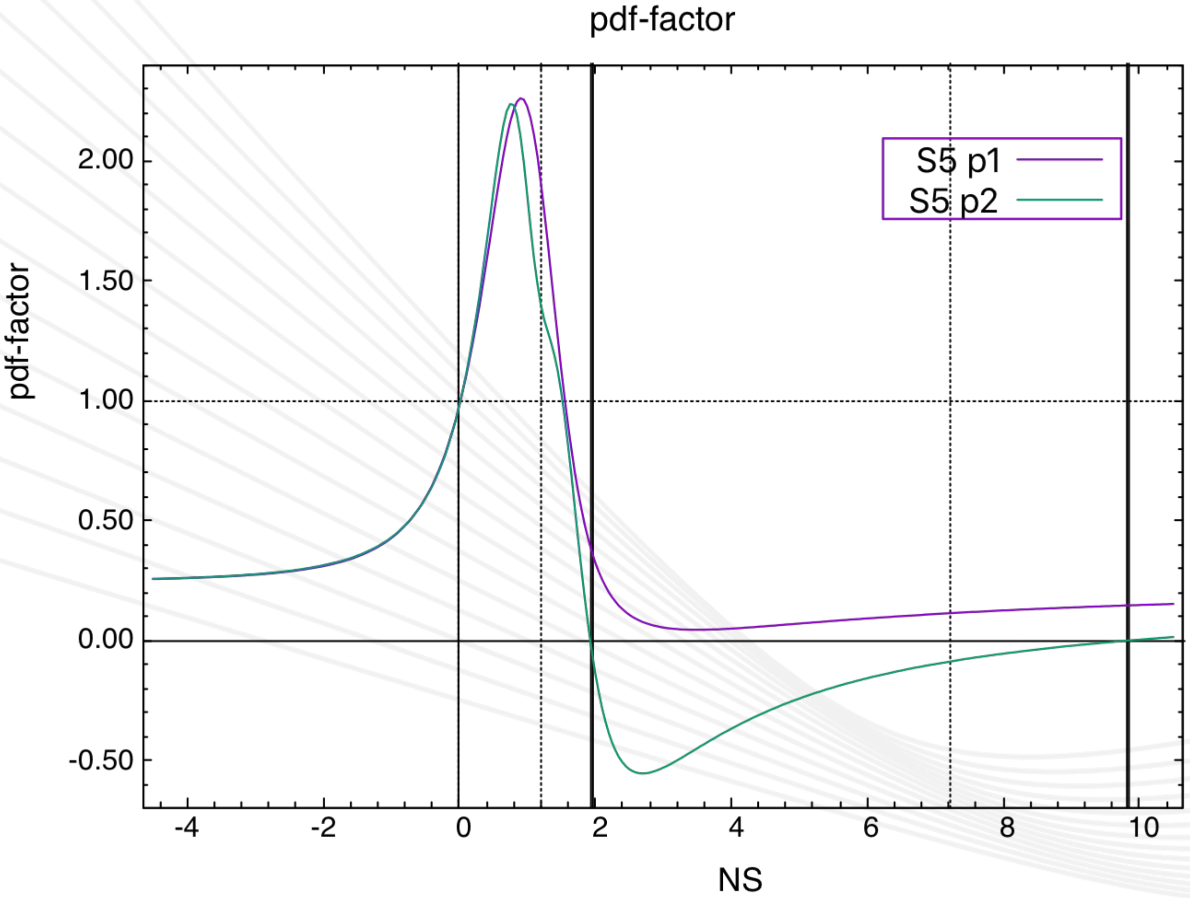
$$c(y) = N(-d_-) + n(d_-)\partial_y \hat{\sigma}(y)$$

Examples of spread and fly arbitrage: S5



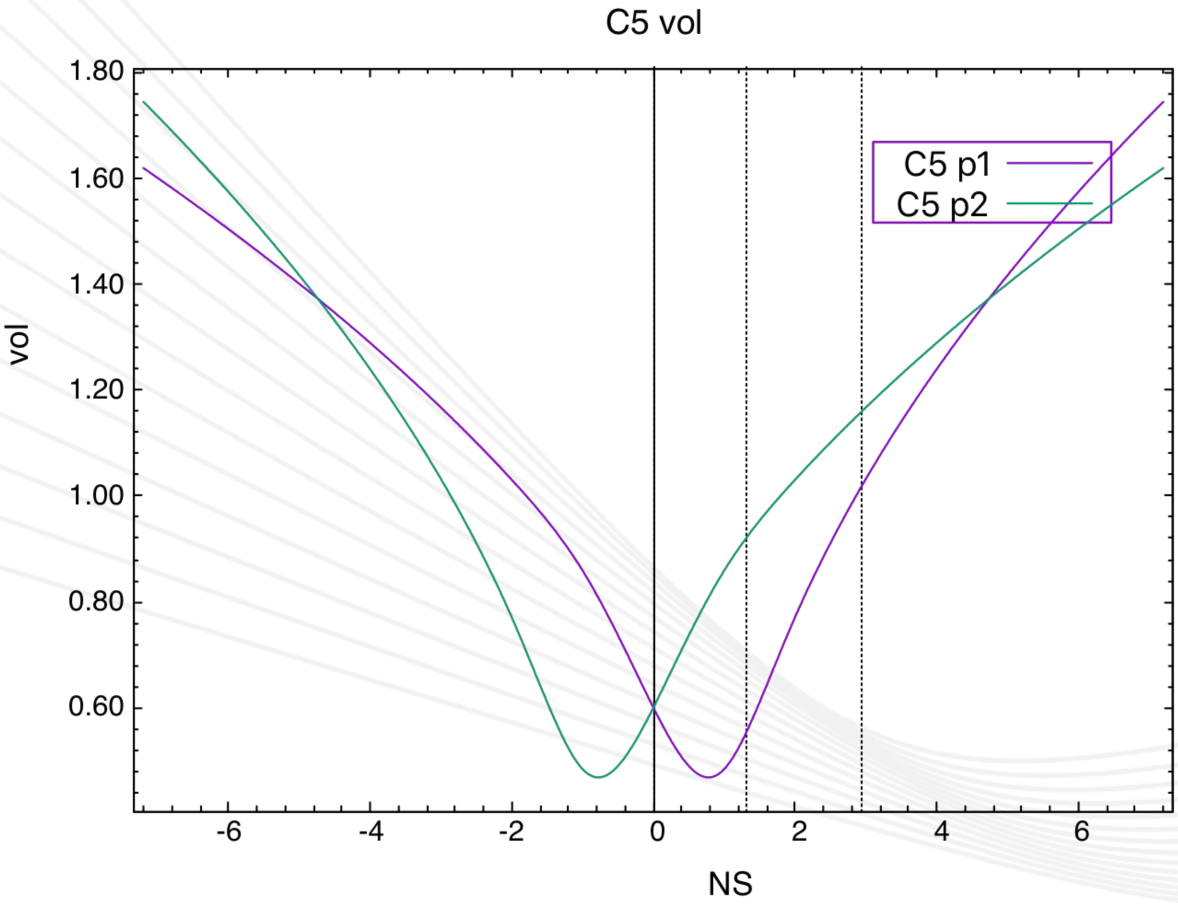
Spread and fly arb come in overlapping regions (if there is spread arb).

Examples of spread and fly arbitrage: S5



g = 1 in BS

Examples of spread and fly arbitrage: C5

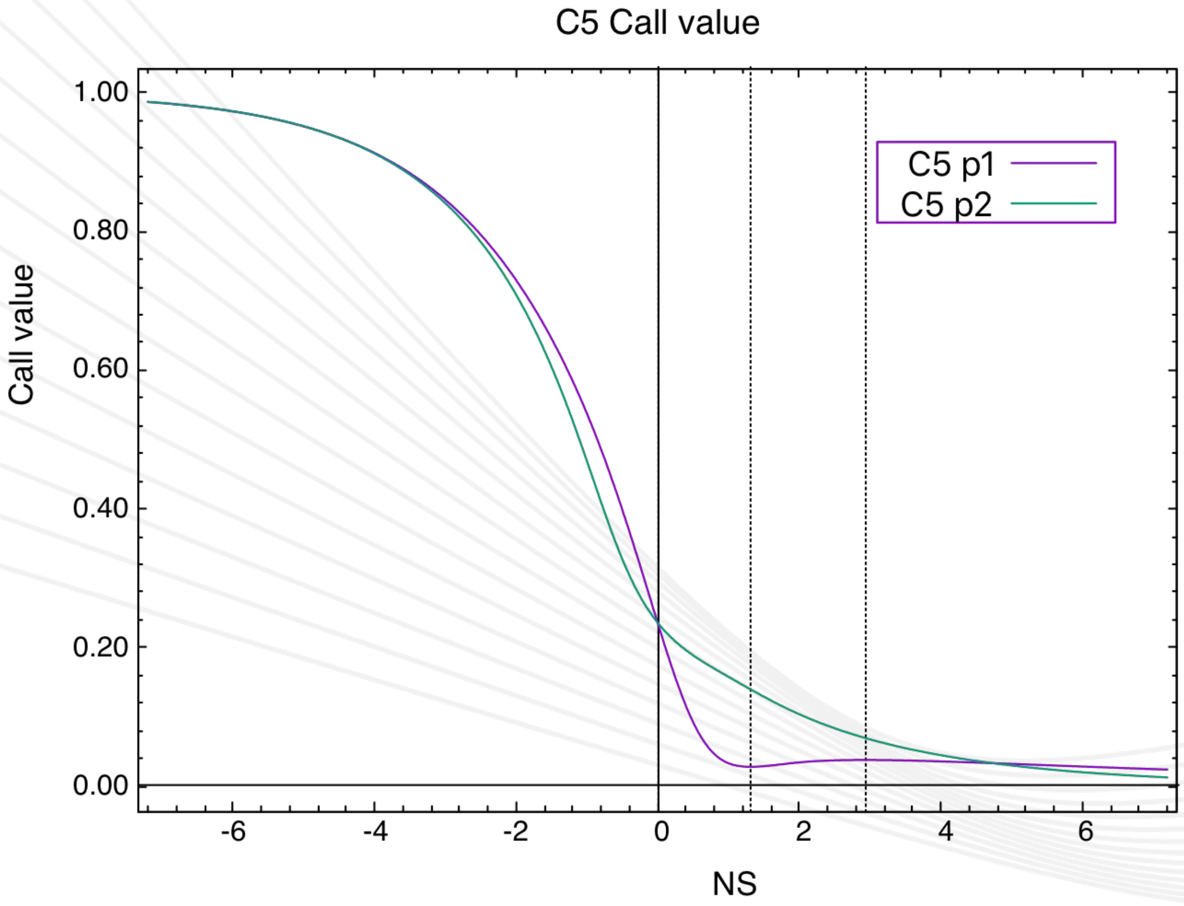


Two "**C5**" curves with perfect mirror symmetry in NS (or y)

Win a Vola T-shirt:

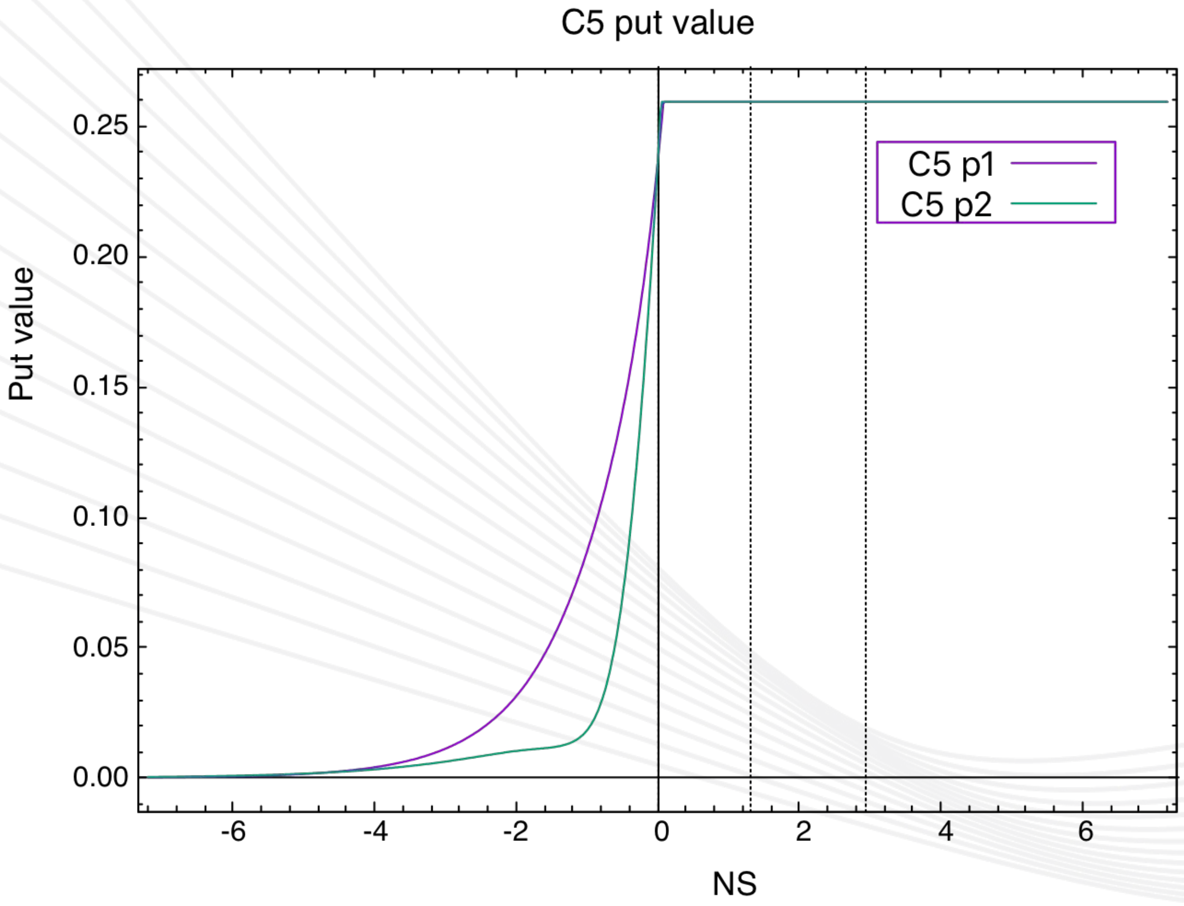
What if any other plot(s) will show a perfect symmetry of some kind?

Examples of spread and fly arbitrage: C5



p1 has call spread arb.

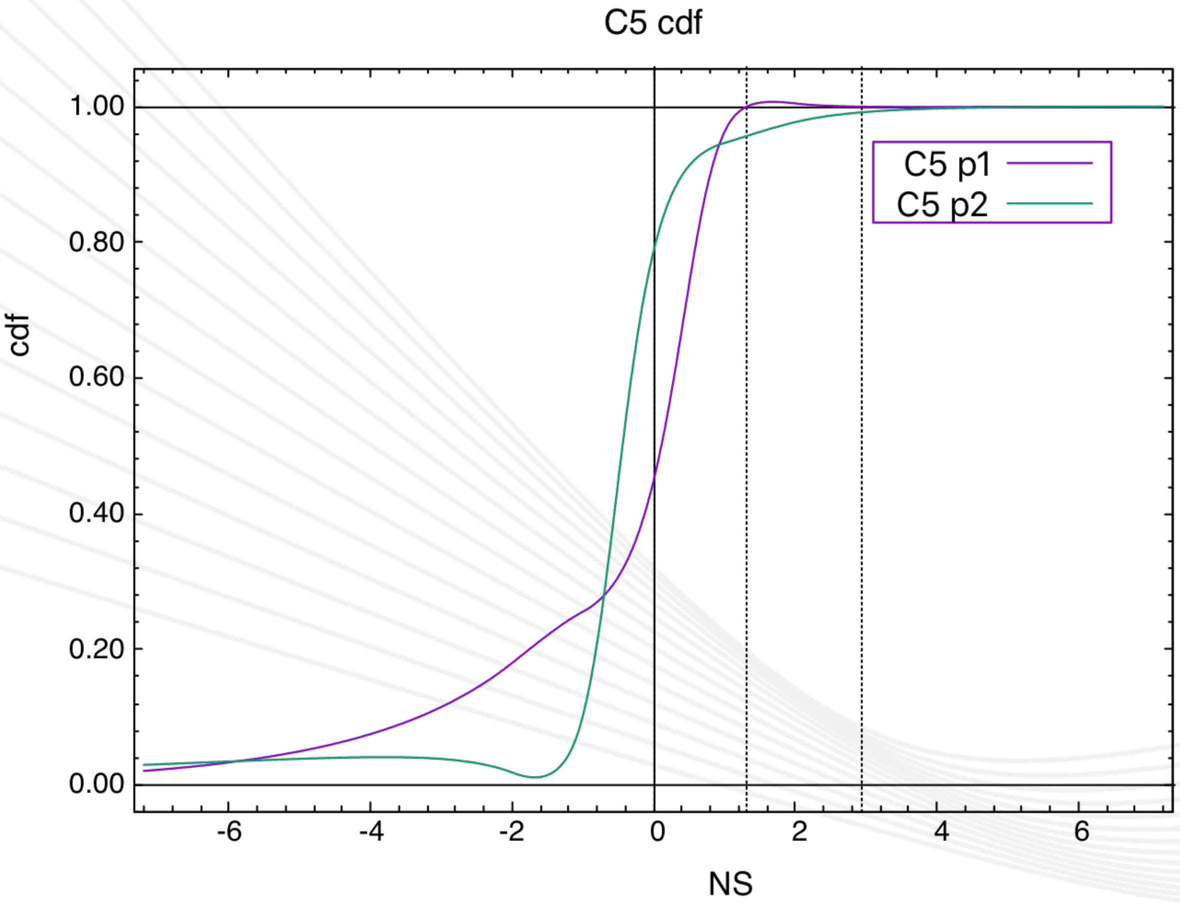
Examples of spread and fly arbitrage: C5



p2 does NOT have put spread arb!

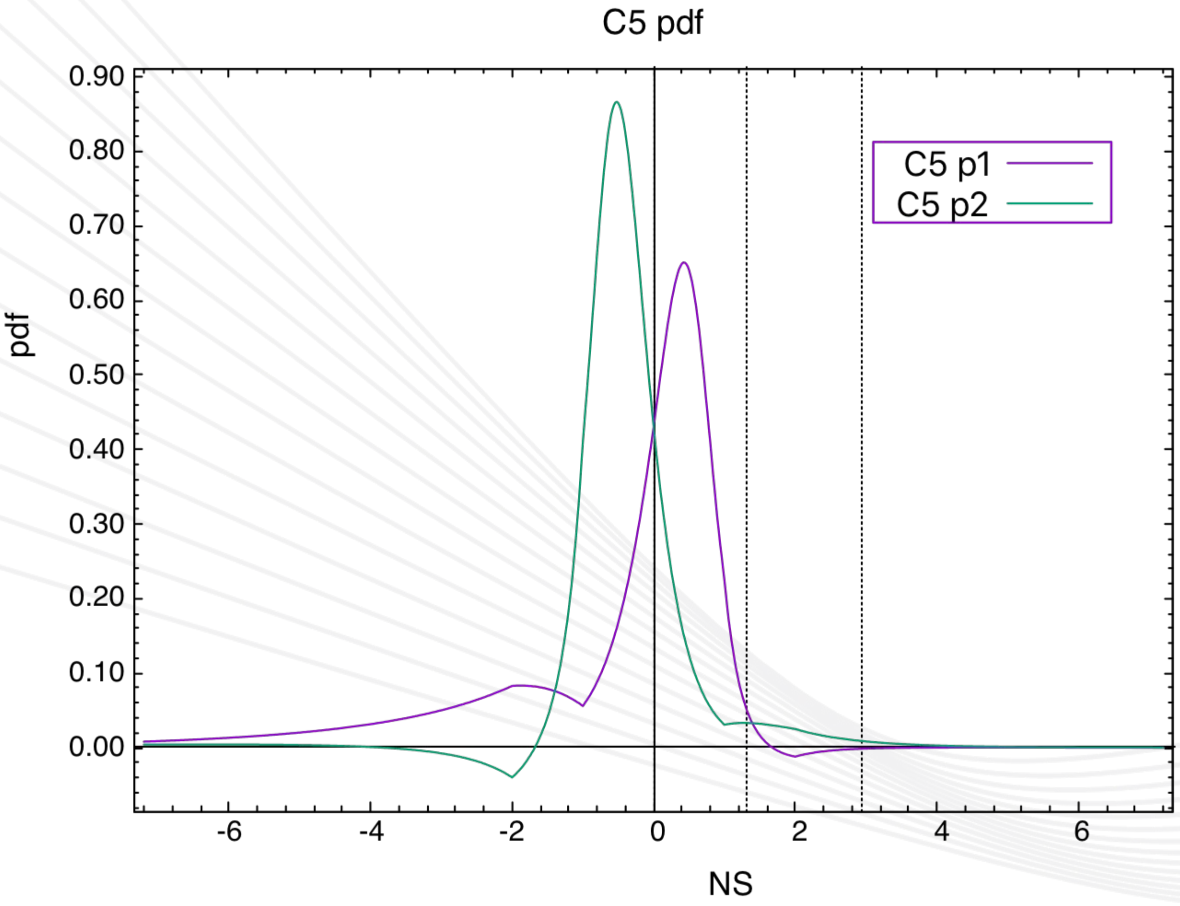
(Puts were cut off at top...)

Examples of spread and fly arbitrage: C5



Indeed, only p1 has (call) spread arbitrage!

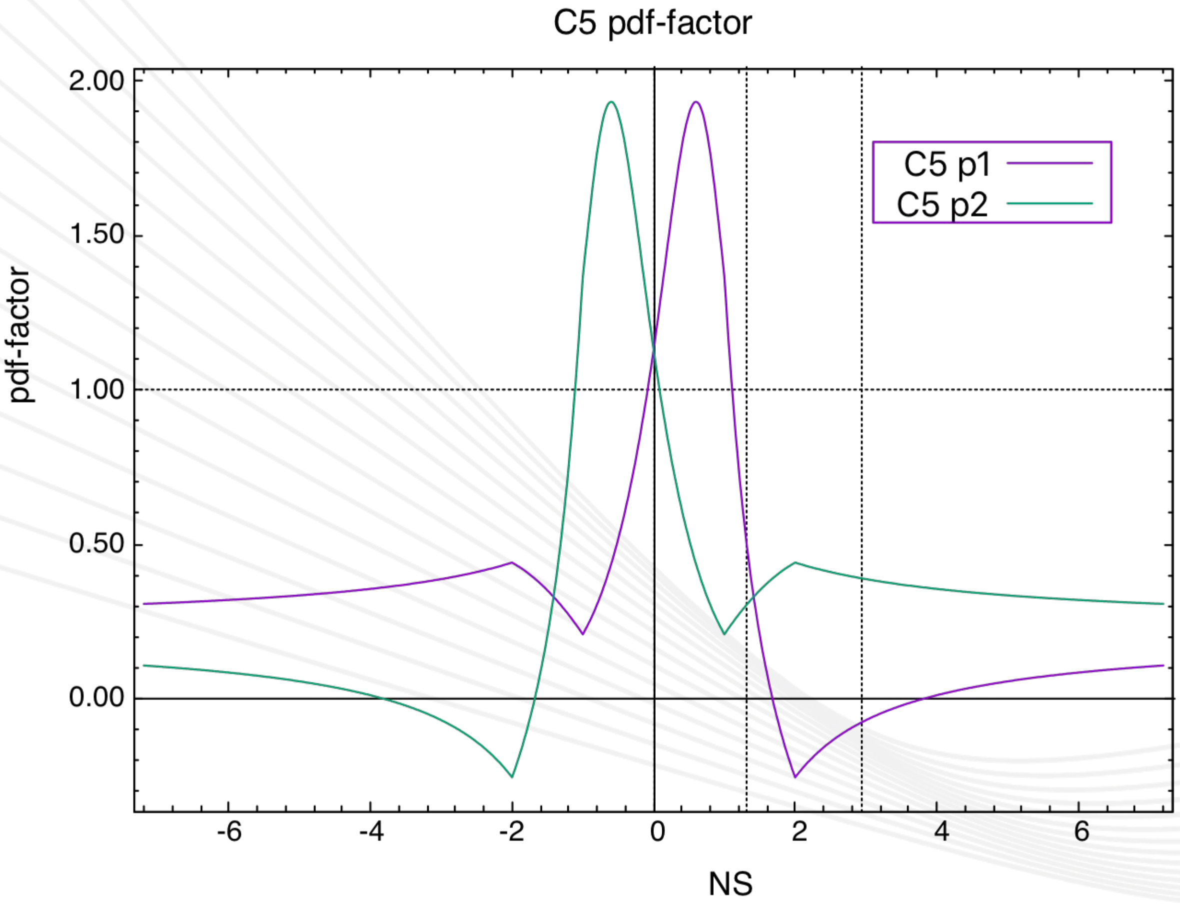
Examples of spread and fly arbitrage: C5



Both have fly arbitrage, but not symmetrically.

Why not symmetric ??

Examples of spread and fly arbitrage: C5

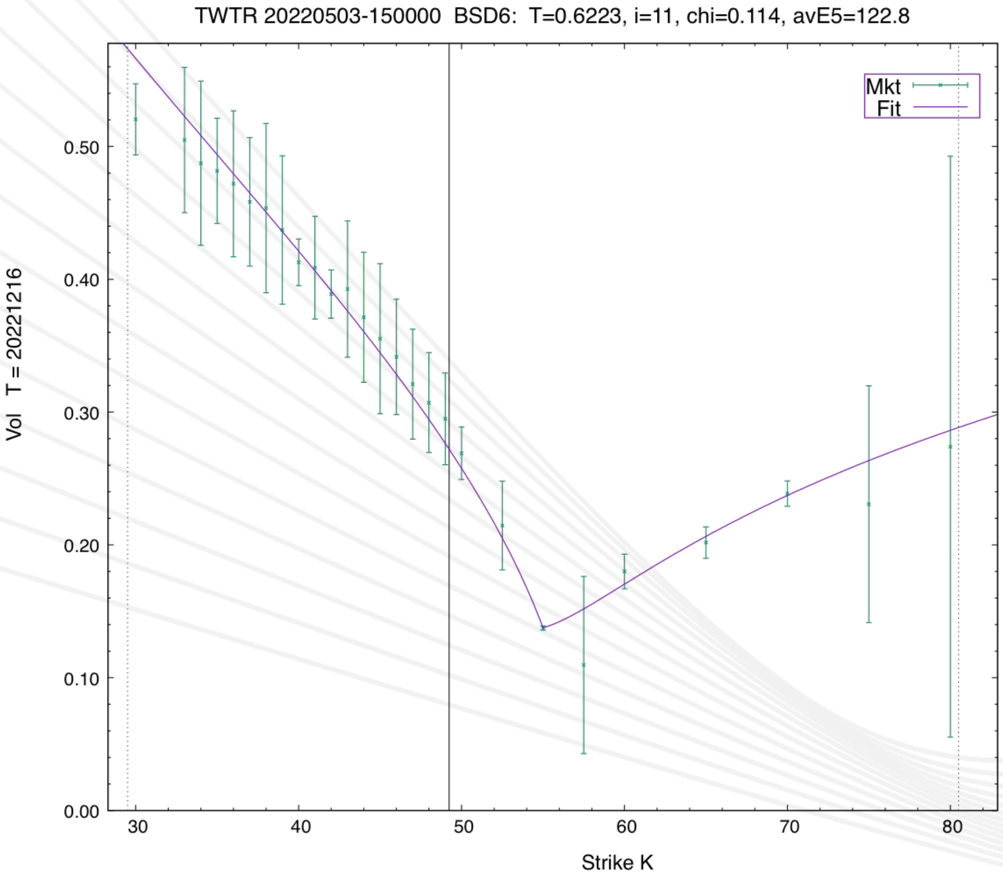


Finally, the pdf-**factors** are symmetric!

The pdfs are not, because of ... ?

(pdfs have same fly arb **regions**, but not fly arb **amounts**...)

Does the density have to be continuous?



Recall: $\hat{\sigma}(y)$ is twice differentiable **except for discrete points**, in general.

Correspond to delta- functions in the density, hence vol slope discontinuities! (Must be positive mass...)

Financially relevant, eg:

- Take-over for cash
- Currency pegs

The slope discontinuity is proportional to the probability of the cash take-over happening at the take-over price!

$$c(y) = N(-d_-) + n(d_-)\partial_y \hat{\sigma}(y)$$

Vol Curves, PDFs, CDFs, Local Vols:

- Good vol curves are a "neat" way to think about (strike)-arbitrage, implied and cum densities, etc.
  - And useful even if there is arbitrage, e.g. the cdf always goes to 1 for large strikes even if there is (massive) arbitrage…
- But there is more… extending the good curves to a good surface, we have eg:
  - LocalVar(T,y) = $\partial_T w(T,y) / g(T,y)$
  - Since Dupire involves only first order T-derivs, T-dependence is less worrisome…
- Working in vol-space with good vol curves provides the fastest and most numerically stable approach to calculating important quantities we care about.

Normalized Arbitrage Metrics 1

- We would like to have dimensionless, normalized arbitrage metrics for butterfly a_K and calendar arb a_T :
  - If they are 0 $\Rightarrow$ no arb. If $\ll 1$, there is very little and hence probably harmless arb.
  - Comparable across terms, underliers, spot-regimes, etc.
  - Ideally can be calculated purely off vol surface, without knowledge of traded T, K, and have well-defined "continuum limit".
- Why do we care?
  - Is intuitive: any trader, quant, or dev can get used to it.
  - Makes quality control of large-scale vol surface fitting infrastructure much easier.
  - Can be used as part of automated vol curve type selection process.

Normalized Arbitrage Metrics 2

- **Butterfly arb**: Obvious — use integral over negative part of density!
  - Average (or max, etc) over terms. Has continuum limit in T-space.
- **Calendar arb**: Look at “rays” $T \rightarrow w(T,y)$ for given log-strike y.
  - Want to take ratio of negative over positive forward variances.
  - De-weight each T, y term as y goes OTM (e.g. vega-weighted).
  - Has continuum limit as more and more T, y are considered.
- Example: Use statistical quality-of-fit criteria plus “penalty factors” based on normalized arbitrage metrics to find best vol curve type for any underlier.
- For details on the metrics, look out for the paper! For now...

Curve Statistics for the Options Universe in 2023

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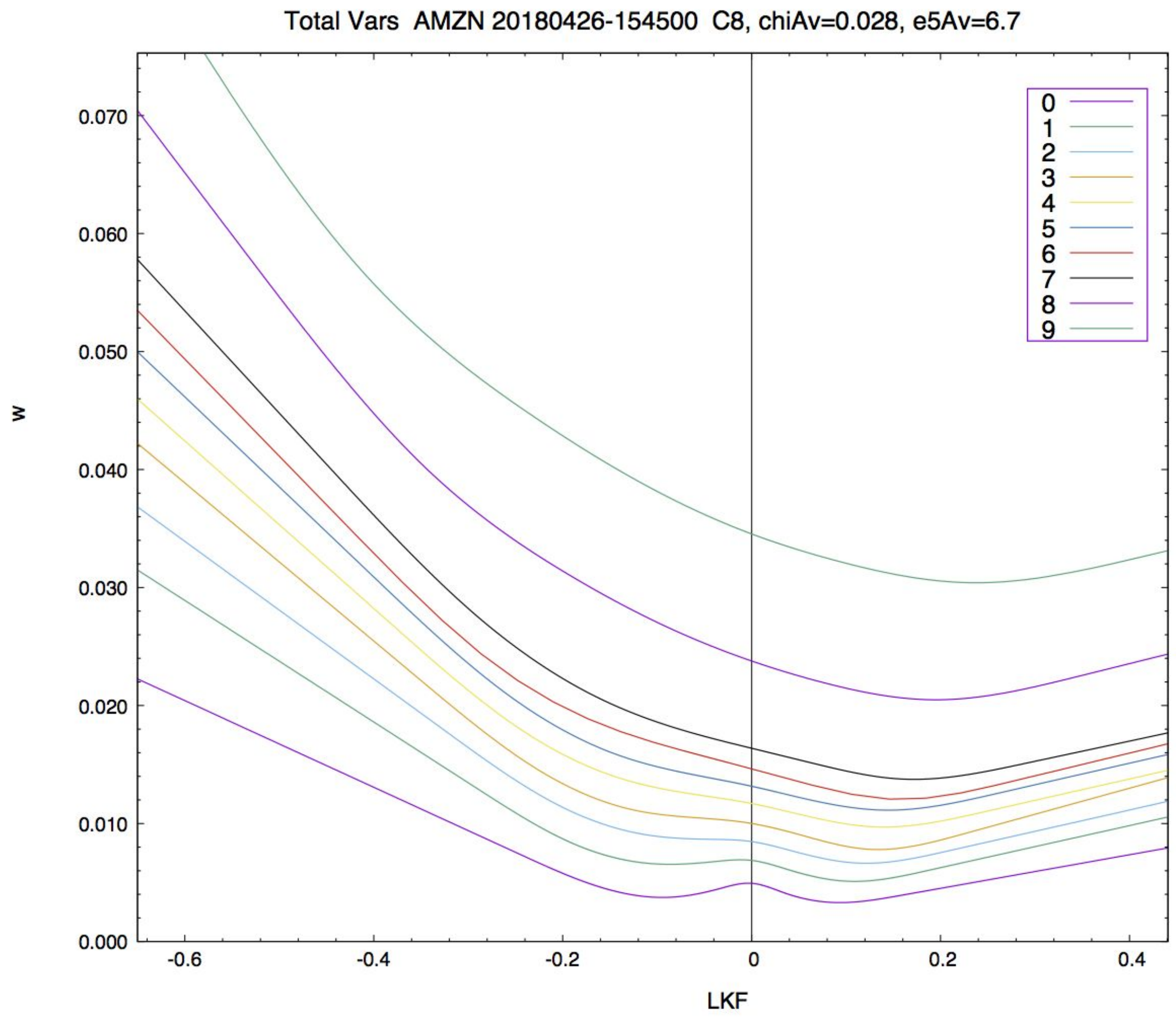
- There are ~5630 names in OPRA (Oct 2023). We find, roughly, for bias-free fits:
  - 4100 (73%) can be fit with S3/SSVI. (S5/SVI: 70 or 1.2%)
  - 650 (11.5%) can be fit with C5.
  - 700 (12.4%) can be fit with C6, C7*, C8* C9*.
  - There are a 70 inverse curves (C6C+) for VIX, VXX, (inverse) leveraged ETFs, low-priced stocks.
  - The remaining 50 (0.9%) names require higher C10 – C16 curves – the most liquid names!
- SPX/SPY/ES require ~16-18 parameters (for some terms) to get bias-free fits of all options down to zero-bids. Some OMMs use 25(+?) params for SPX.
- Big tech names and (other) global indices require 9 –15 params per term.
- There has been a relentless drive towards higher curves, to fit tighter spreads and wider (normalized) strikes ranges.
  - Empirically, roughly (for OPRA universe): $\text{nParams} \approx (\text{nOptions} / 5)^{1/3}$

Some final examples of living dangerously...

- Namely, examples of surfaces close to arbitrage, either calendar or fly.
- In particular, what do the funky vol shapes mean, in terms of the markets expectation about the future?
- These expectations are a lot more specific and sophisticated nowadays than e.g. during the GFC in 2008.



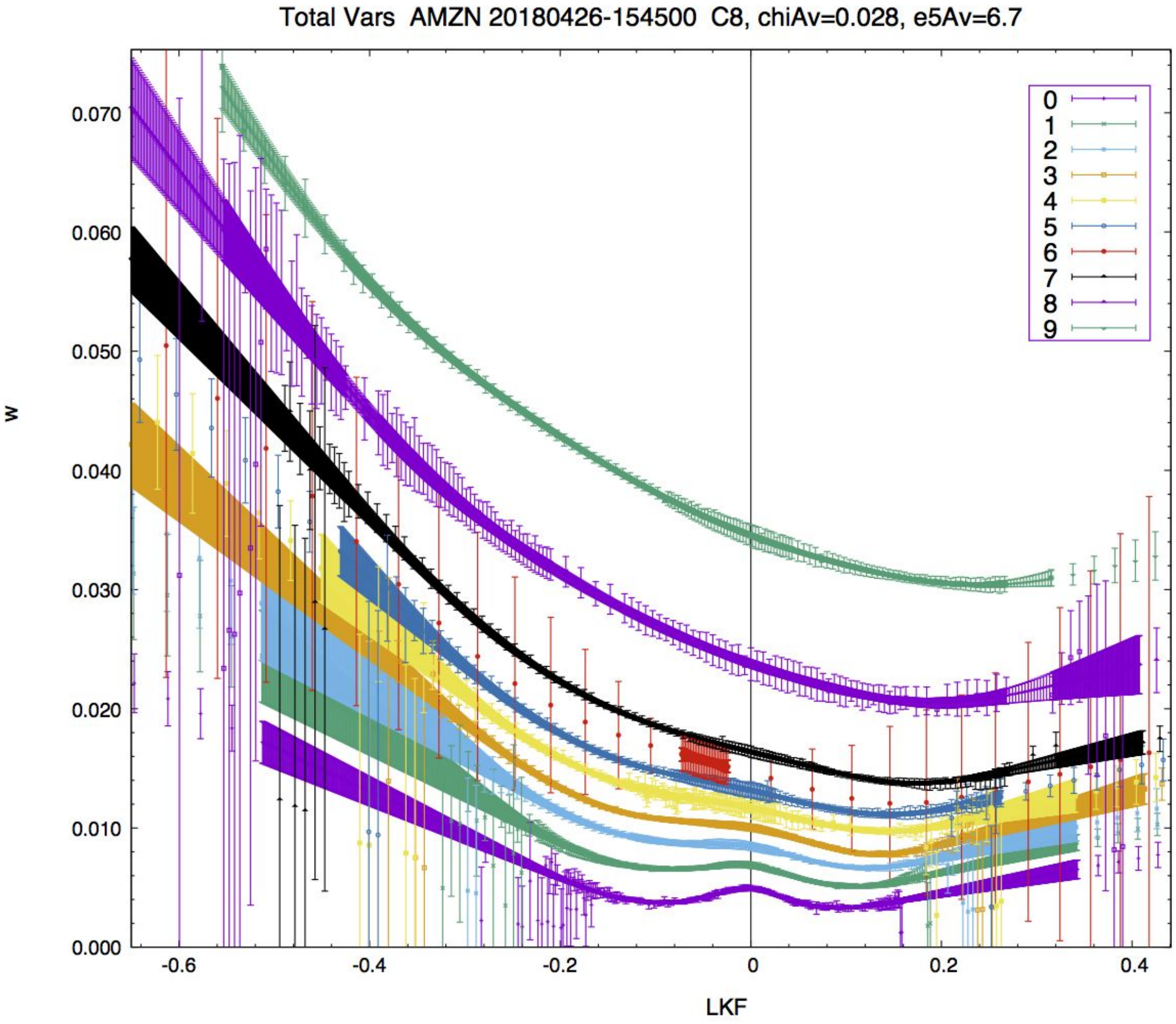
AMZN 2018-04-26 earnings day

C8 total variance plot

First 10 terms

No calendar arbitrage! (Or butterfly…)

Interesting Thursday: Earnings, new weekly listed (**i=6**), etc.



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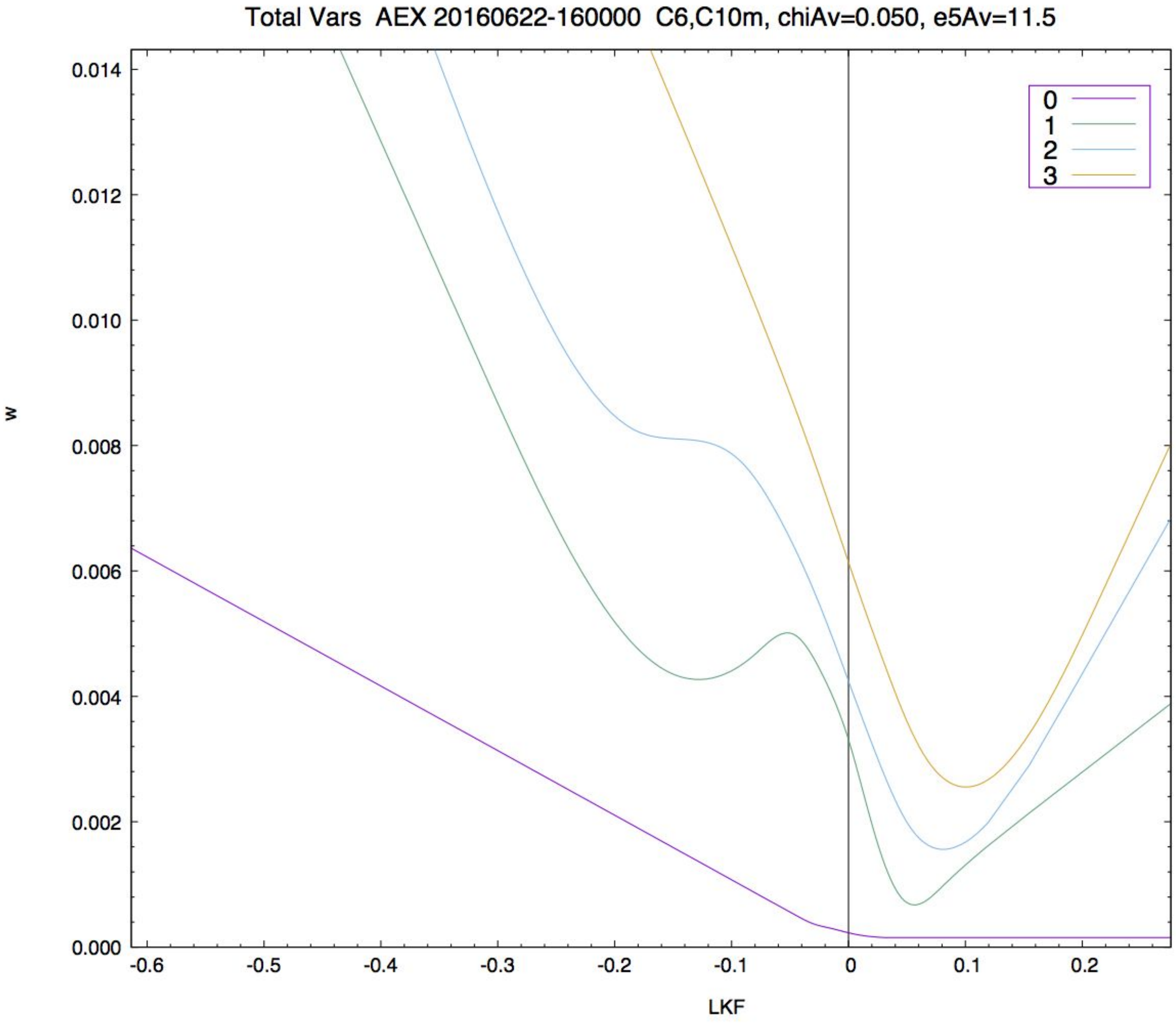
AMZN 2018-04-26
earnings day

C8 total variance plot

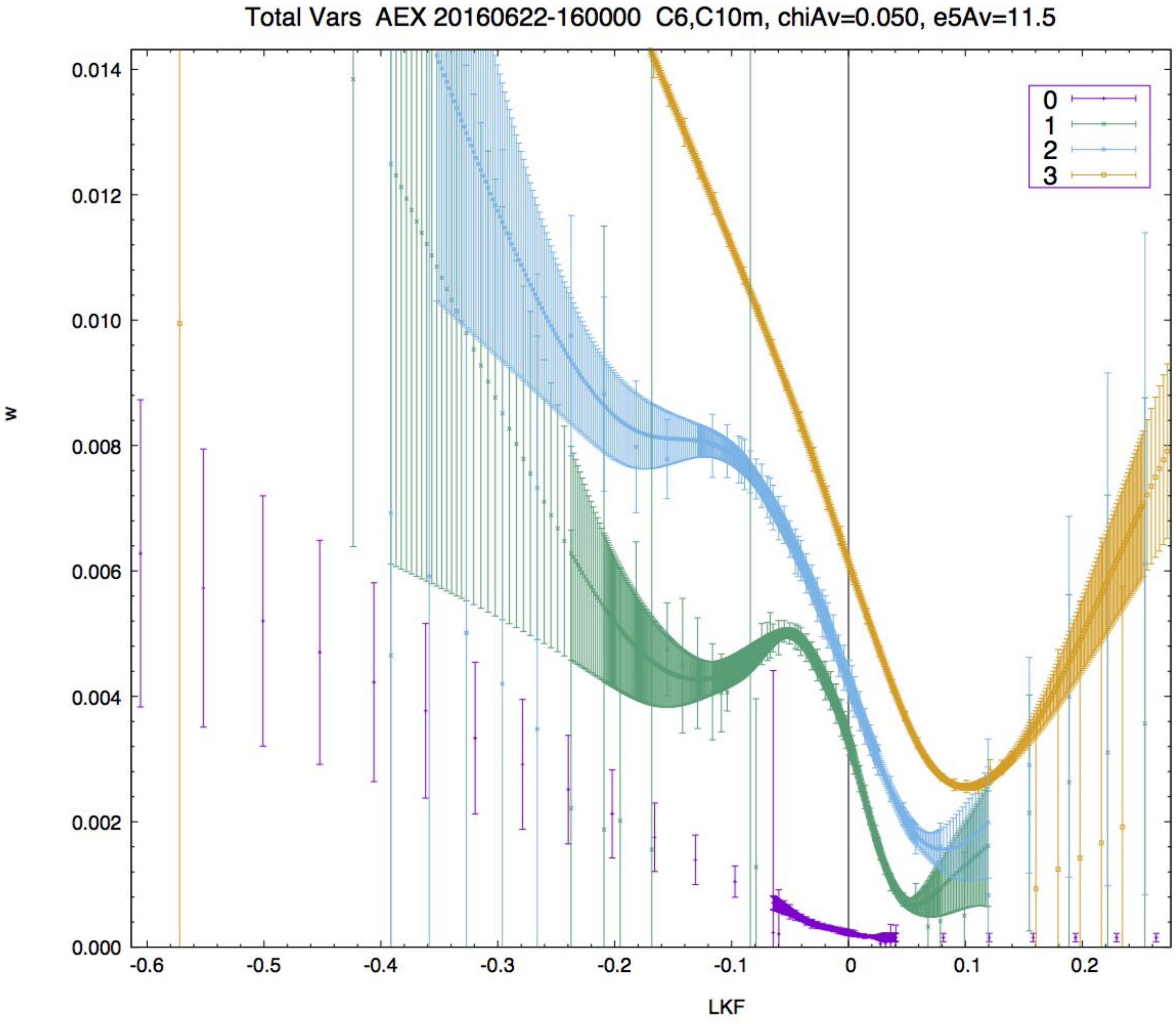
First 10 terms, with errors bars

Interesting Thursday: Earnings, new weekly listed (i=6), etc.



Fitting **AEX** on day before Brexit

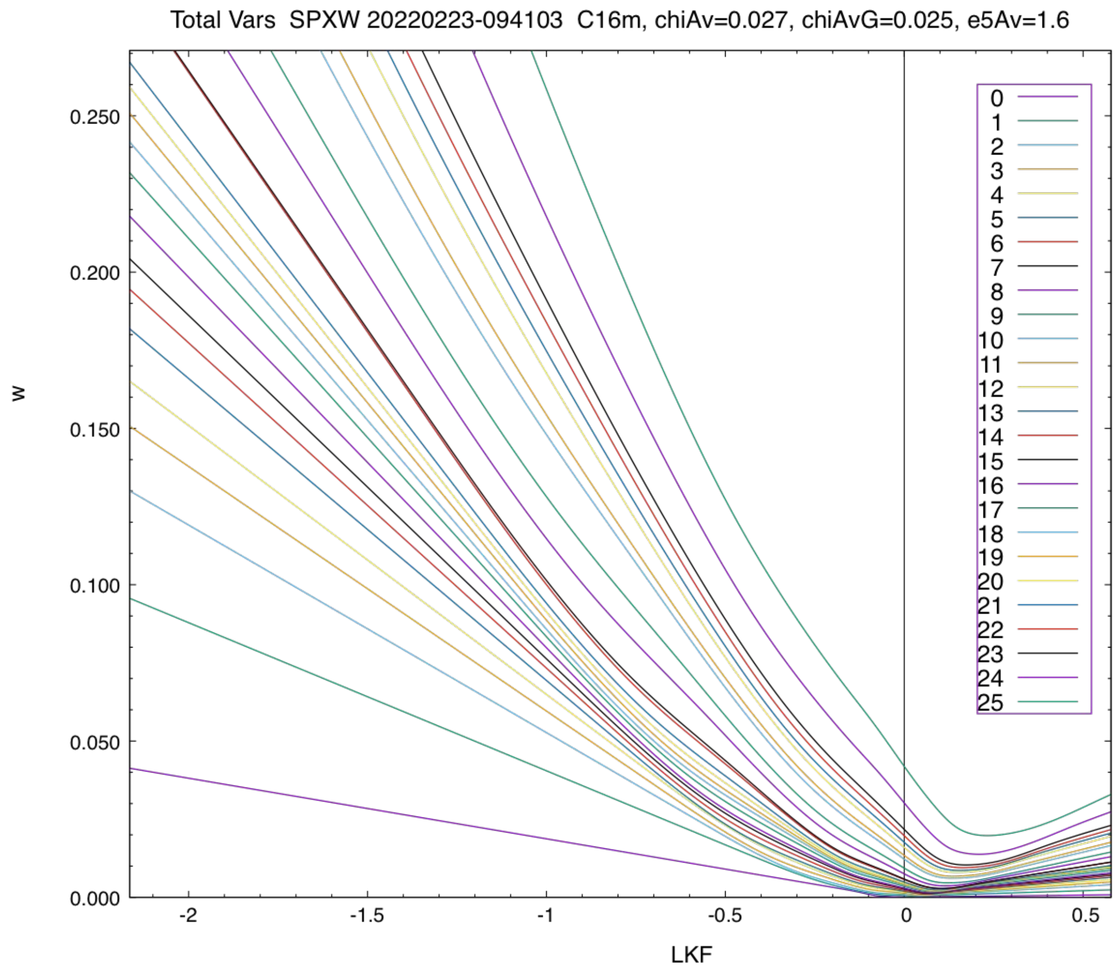
Total Var plot



VOLA DYNAMICS
Analytics for Options Trading

Fitting **AEX** on day before Brexit

Total Var plot
with error bars



SPX 2022-02-23

Day before Ukraine invasion

C16m total variance plot

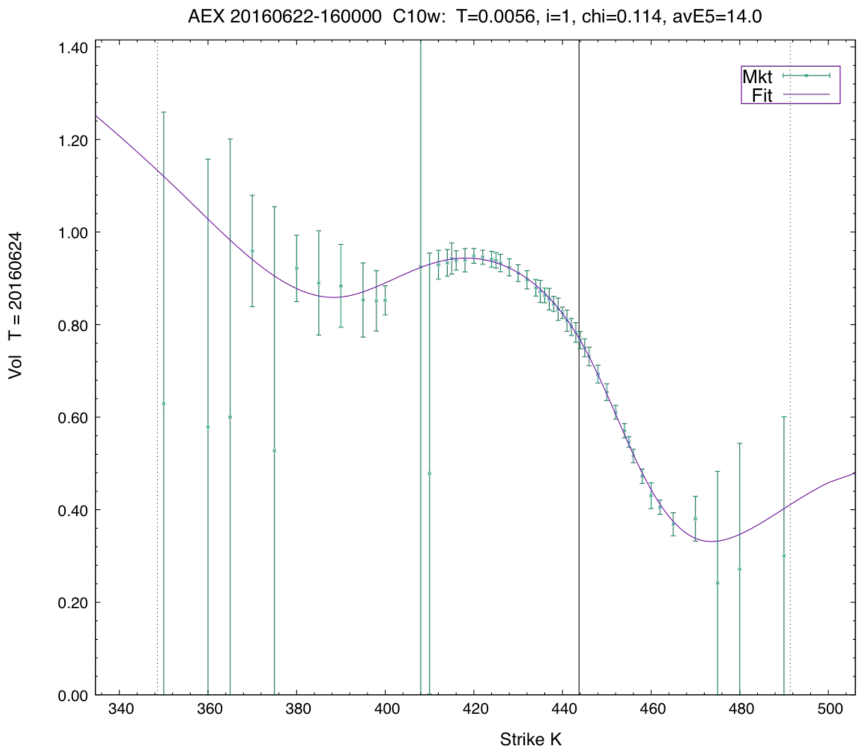
No crossings! (even i=14,15)

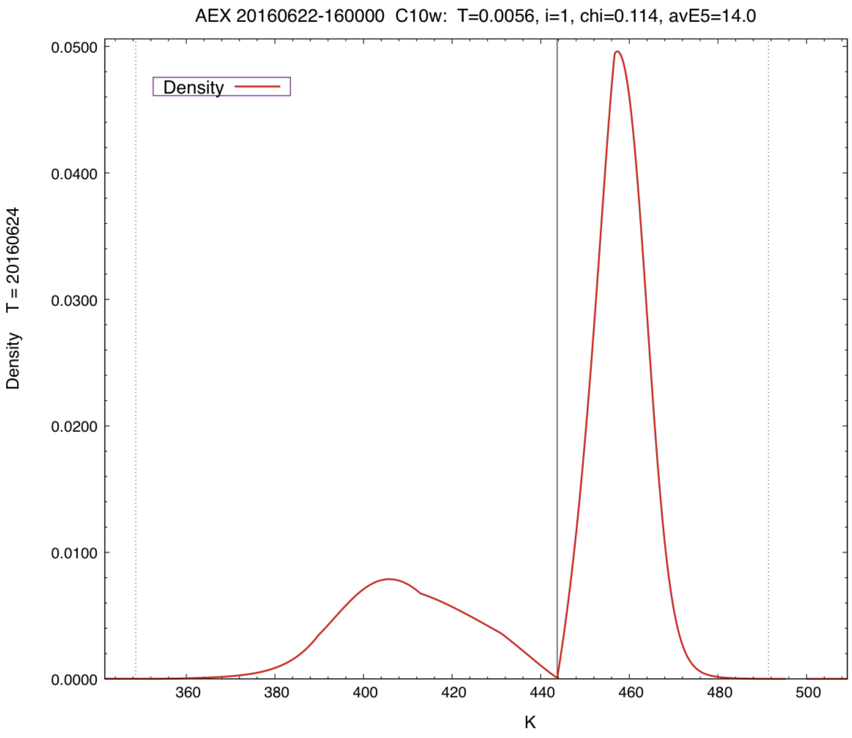
No calendar arb!

Just SPXW for clarity (and harder…)

AEX on day before Brexit vote:

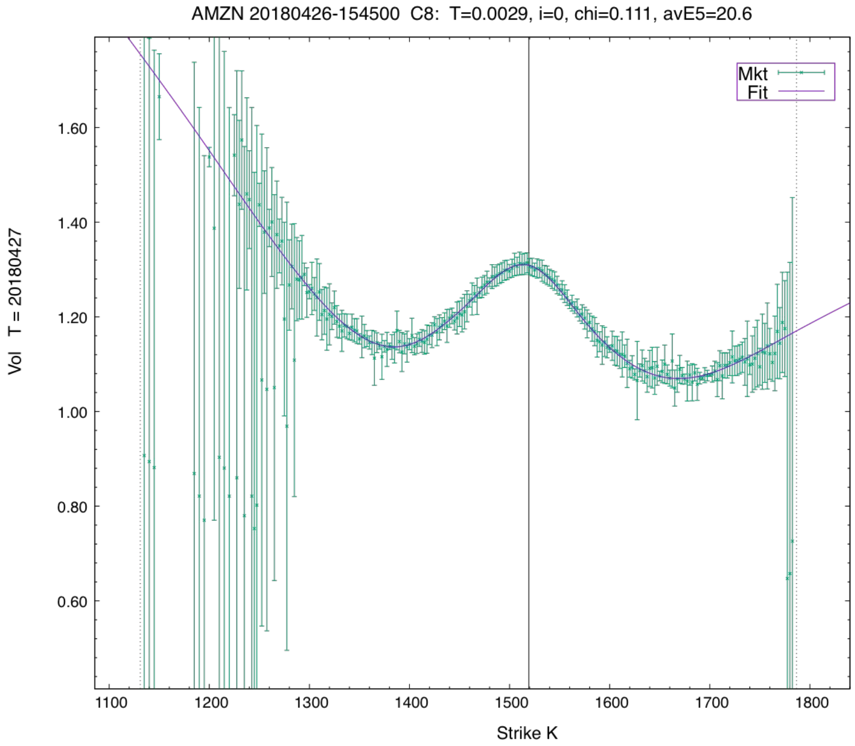
T=2d, vols and implied density

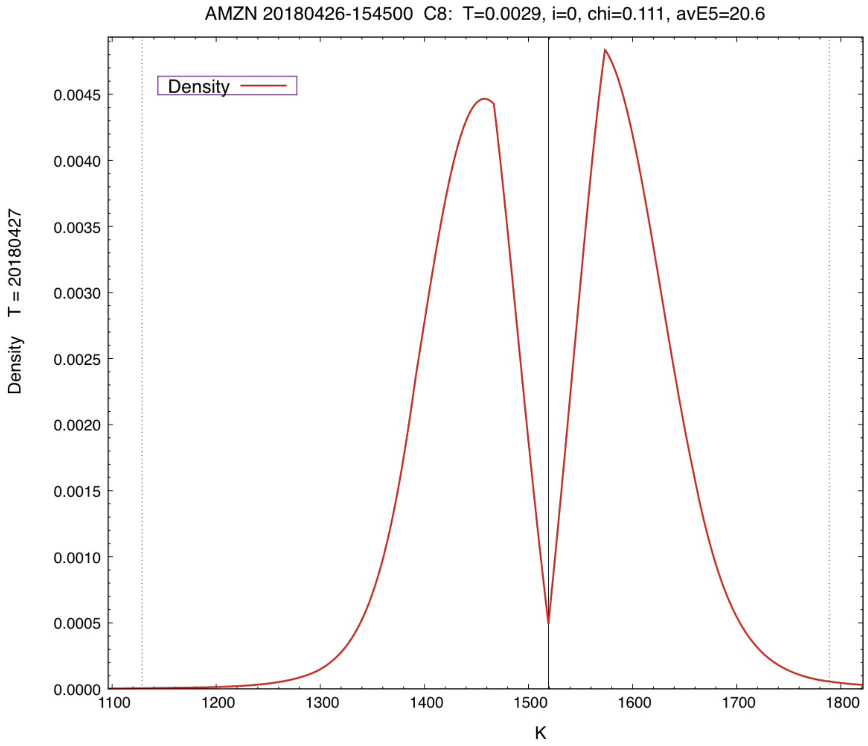




AMZN 2018-04-26 earnings day:

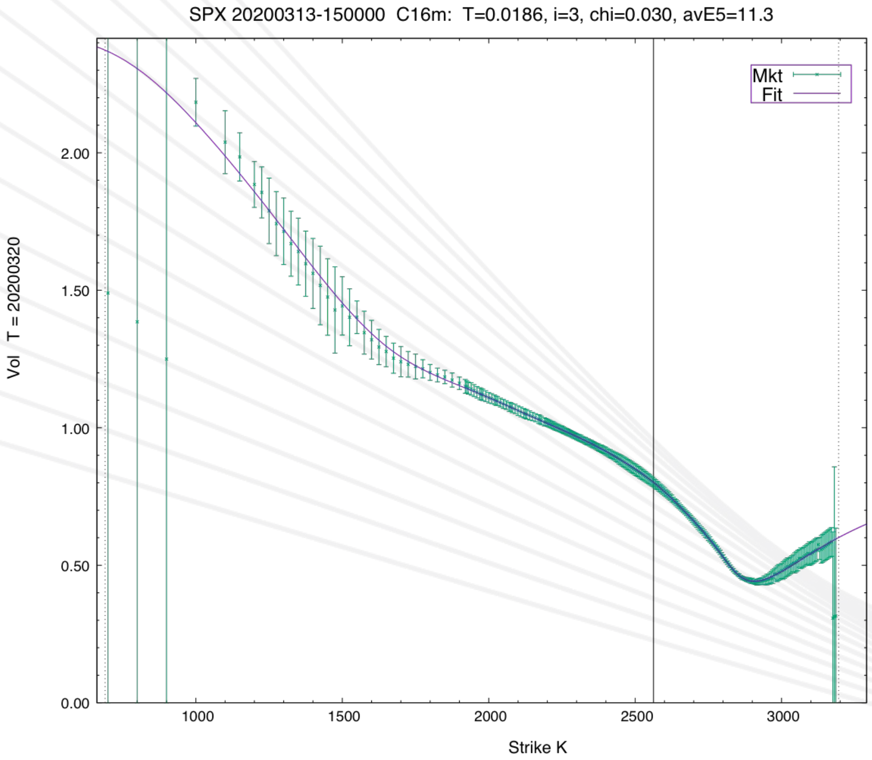
T=1d, vols and implied density

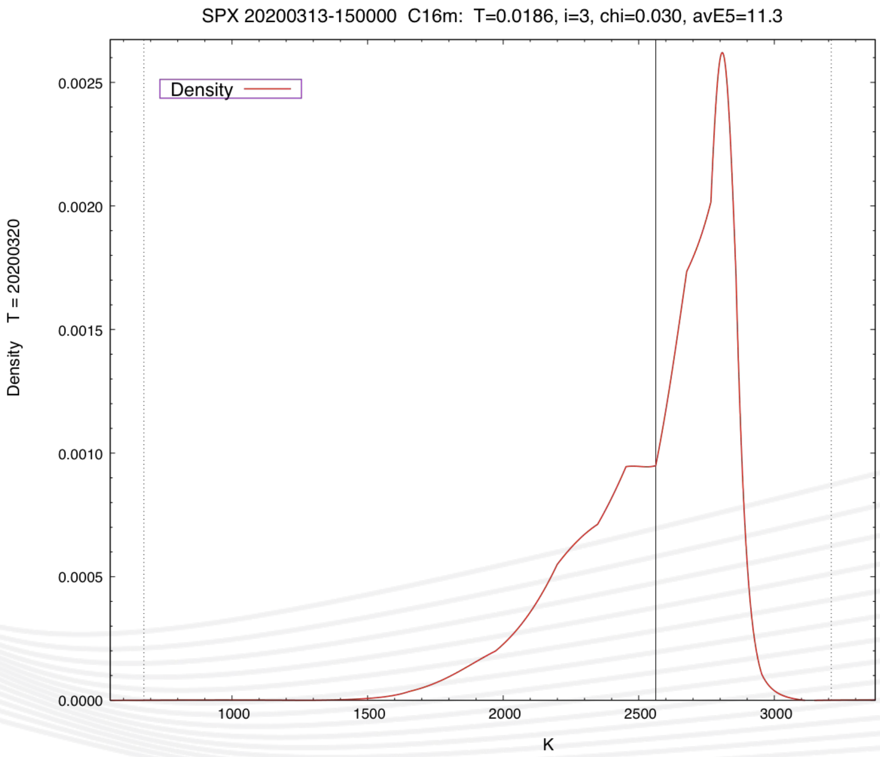




SPX 2020-03-13: During covid crash

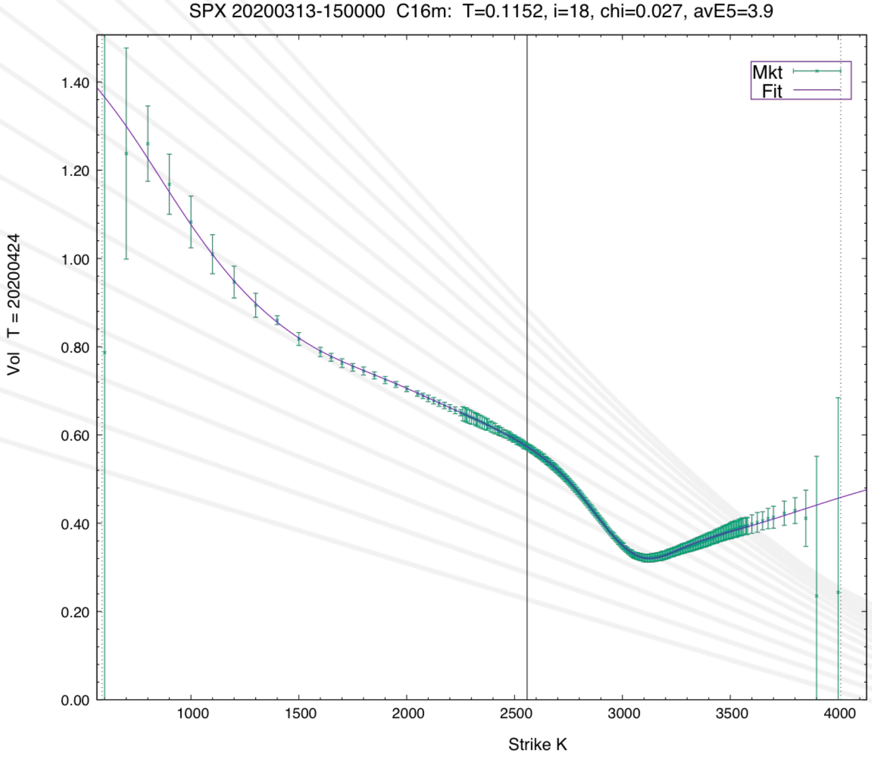
T=1w, vols and implied density

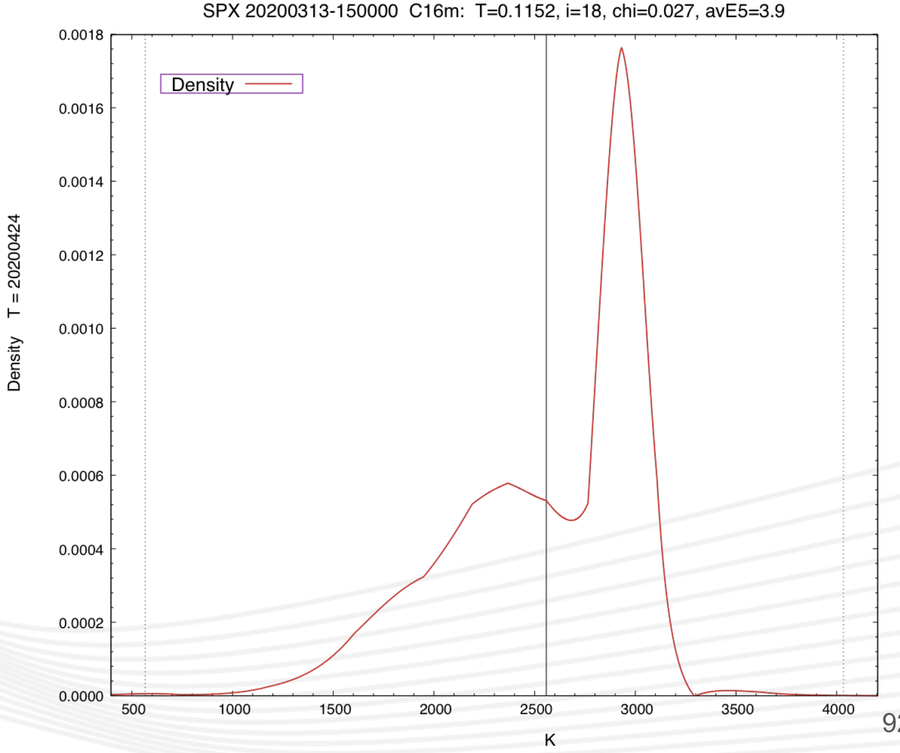




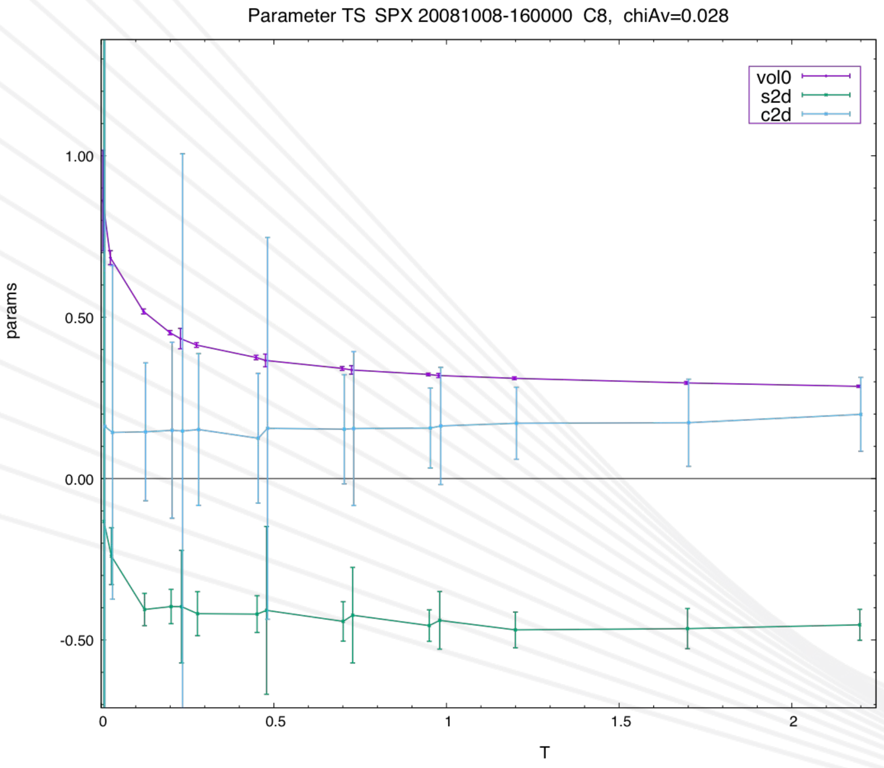
SPX 2020-03-13: During covid crash

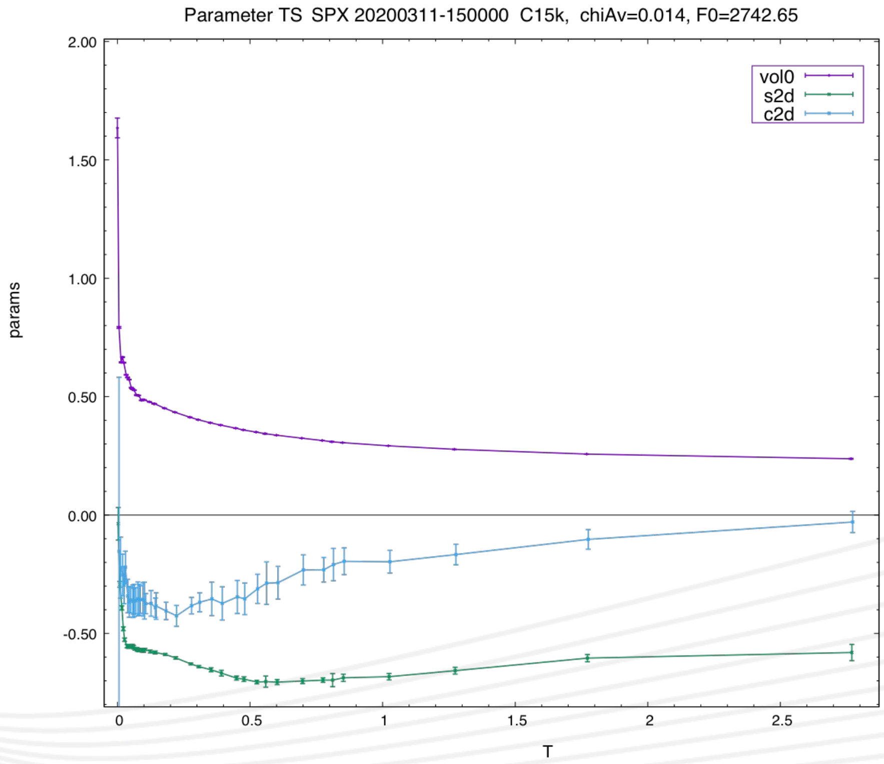
T=6w, vols and implied density



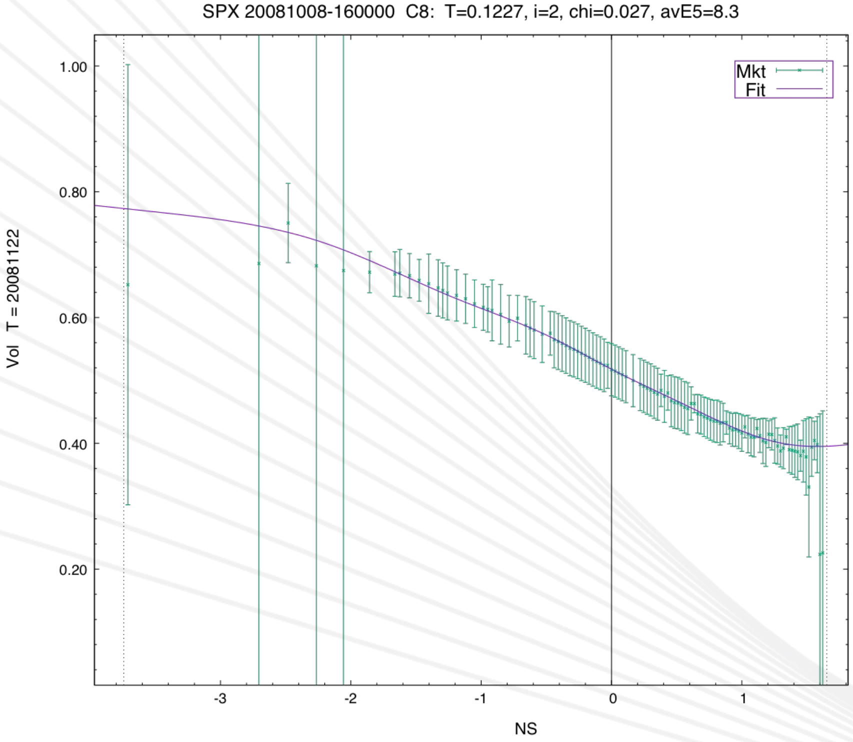


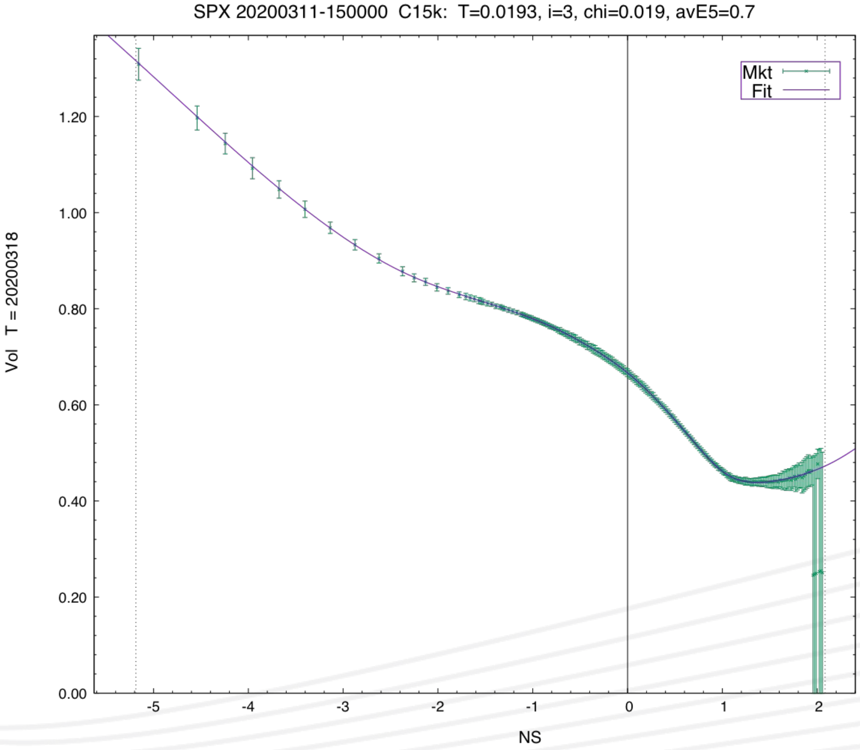
Parameter TS: 2008 versus 2020





Vol Skews: 2008 versus 2020





SPX Spot-Vol Dynamics: Basics

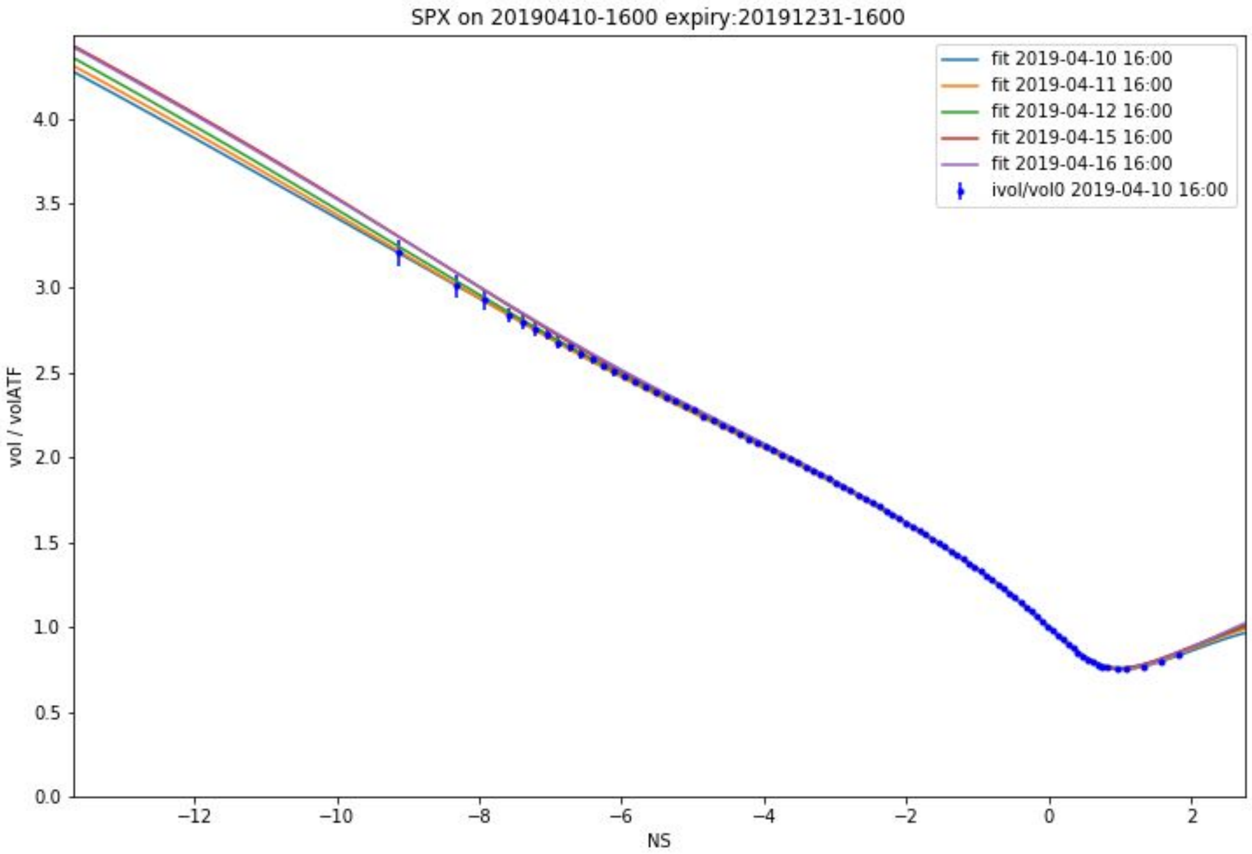
- Shape (by NS or Δ) is much more stable than overall vol level (vol0 aka ATF vol).
  - Sticky-strike or sticky-delta vol dynamics does not hold at all (for equities for 15y+).
- ATF vol dynamics is very well described by one dimensionless number, SSR aka vol sensitivity aka super-skew, which is the ratio of vol0-path & skew slopes.
  - Even when SSR = 1, i.e. sticky-strike around ATF, is the behavior in the wings usually much better described by fixed NS-shape than by sticky-strike.
- Very simple dynamics in terms of NS vol parameters (e.g. just ATF vol), gives complicated vol-by-strike dynamics, that actually describes market moves.
  - It also gives the correct adjusted (aka smart aka skew) deltas and gammas (see LinkedIn article).

SPX Spot-Vol Dynamics: Then and Now

VOLA DYNAMICS

Analytics for Options Trading

- In the olden days:
  - Virtually no shape dynamics.
  - Overall vol level dynamics described very well by one SSR with little term-structure (TS).
  - 1 < SSR < 2, with 2 reached only on big down days. Typical value SSR=1.3.
- Nowadays:
  - There is often term-structure in SSR, with SSR(T>1y) closer to typical values.
  - There is occasionally, e.g. on some big down days, shape dynamics, eg in c2.
  - SSR > 2 and SSR < 1 can happen, on short end.
  - Some horizon dependence (1min, 5min, etc), including intraday vs overnight differences.
  - More “fluctuations”, in path-dependent manner (cf. Guyon), around typical values.
  - Open Q: How strong is path-dependency effect relative to levels set by “SSR regime” ?

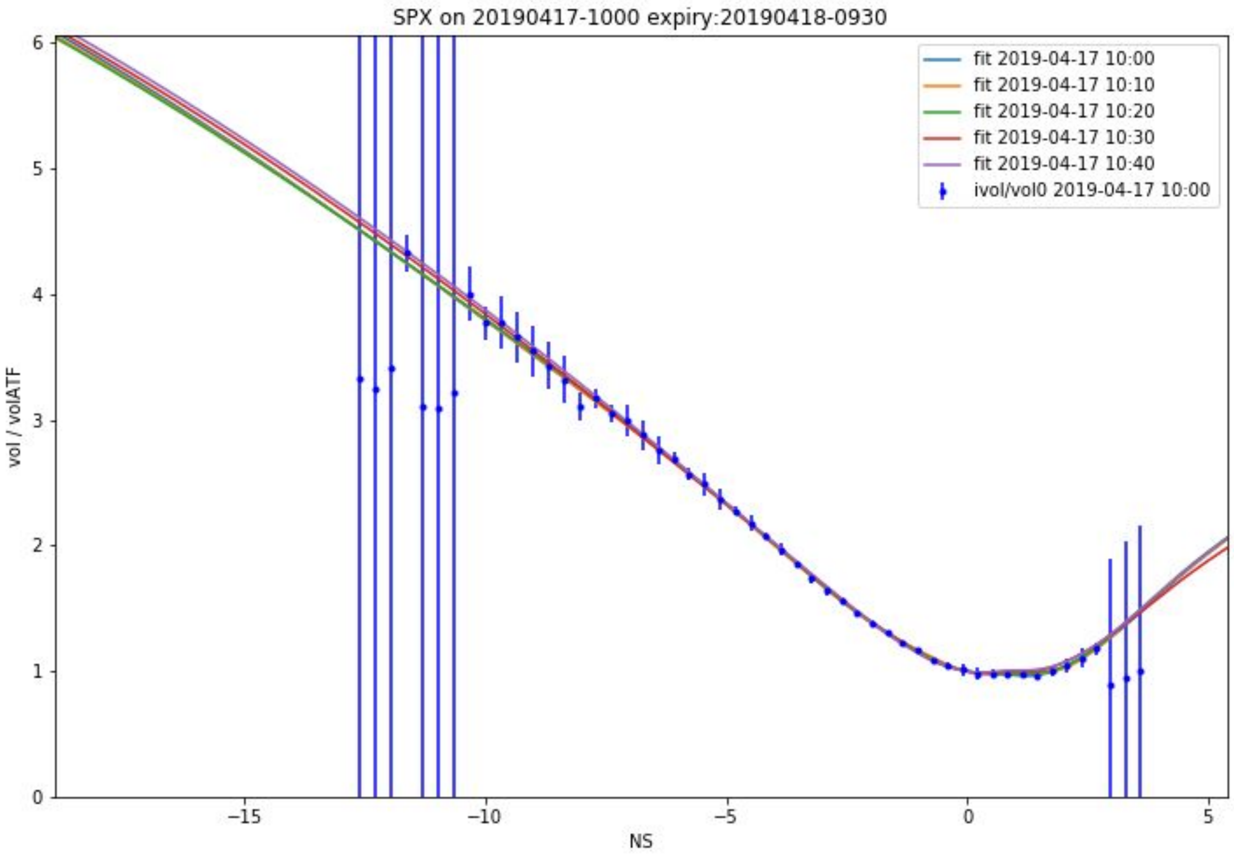


Stability of NS Shape

SPX 20190410 T = 9m

Shape stable over many days, while underlier moves around.

Also, no floppy wings!

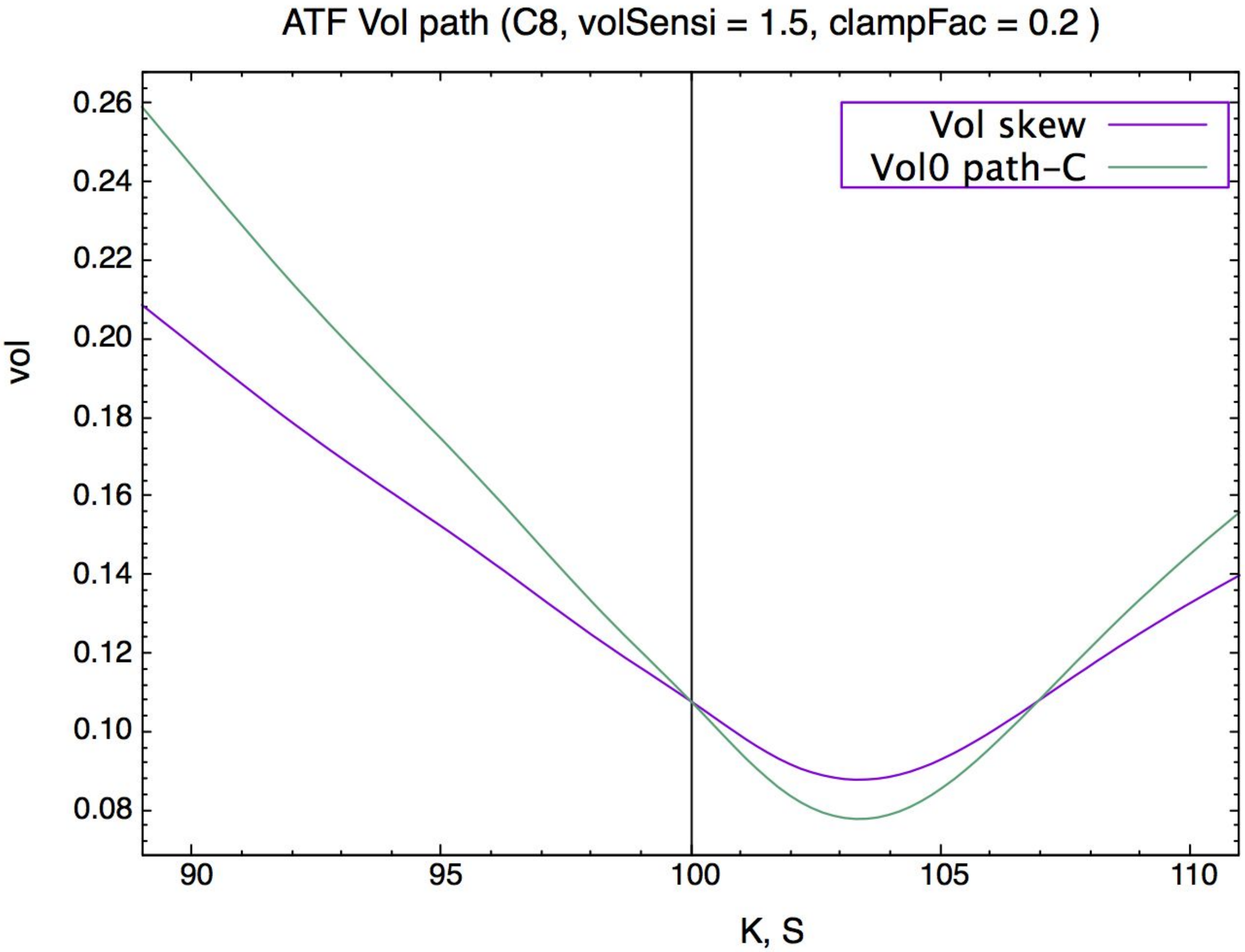


Stability of NS Shape

SPX 20190410 T = 1d

Shape stable even on last day

Also, no floppy wings!



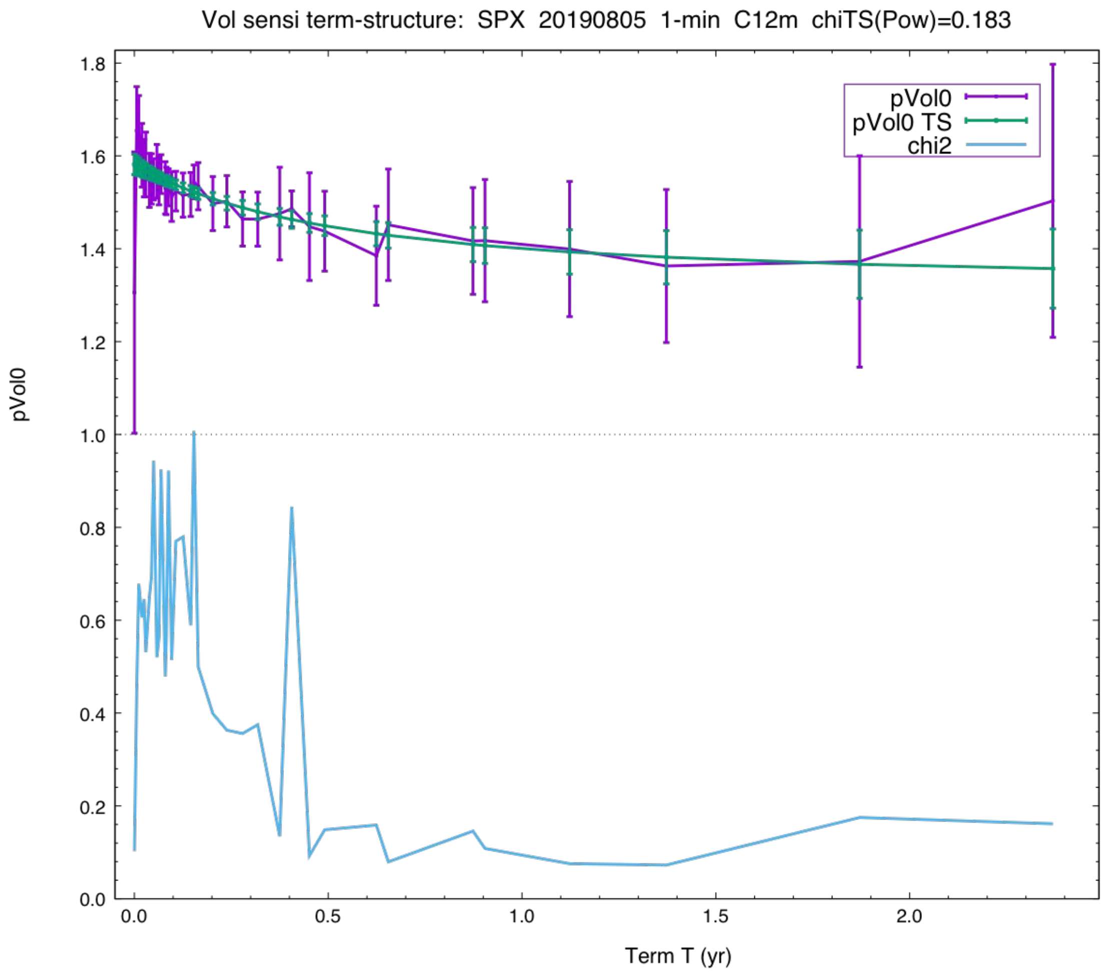
Spot-Vol Dynamics

ATF “vol path”

SSR = 1.5

“Along curve”

No clamps

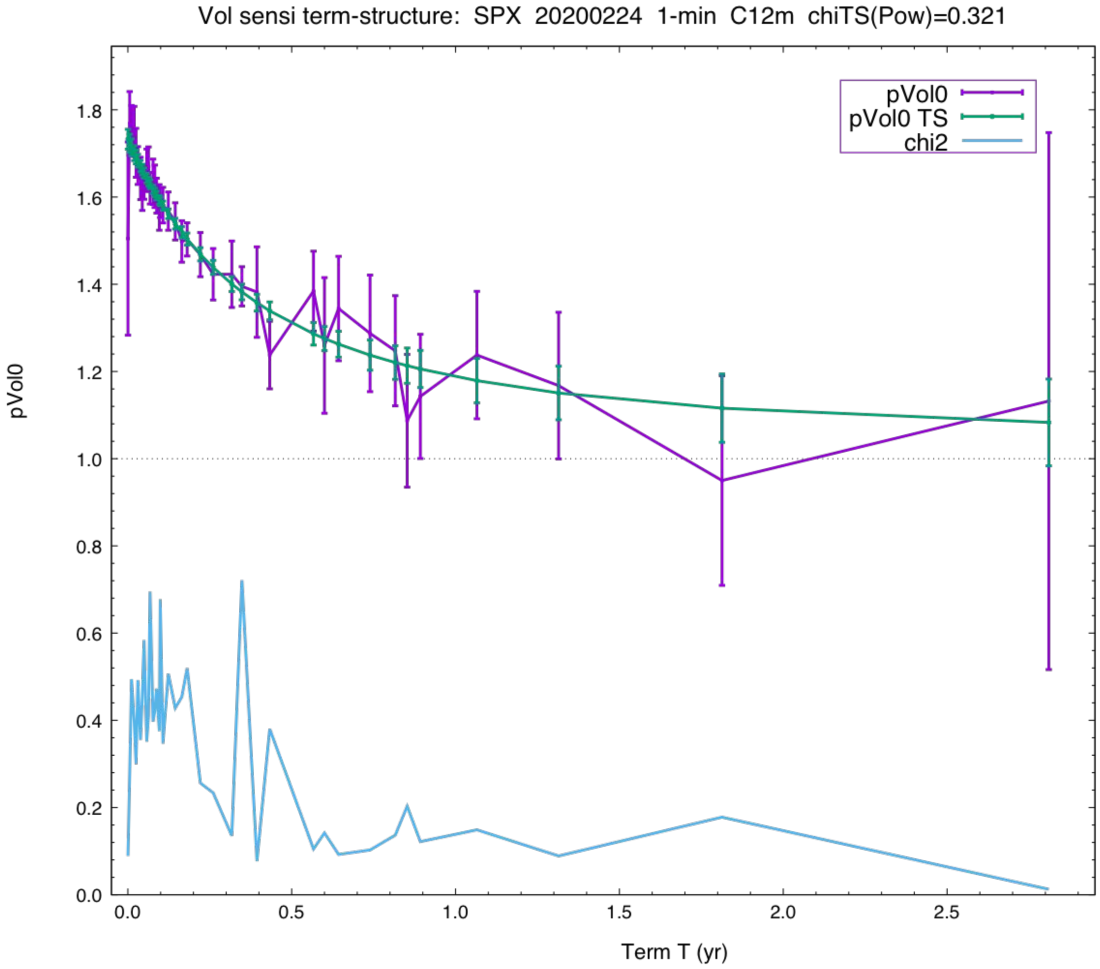


VOLA DYNAMICS
Analytics for Options Trading

SPX 20190805

Vol sensitivity (SSR) term-structure

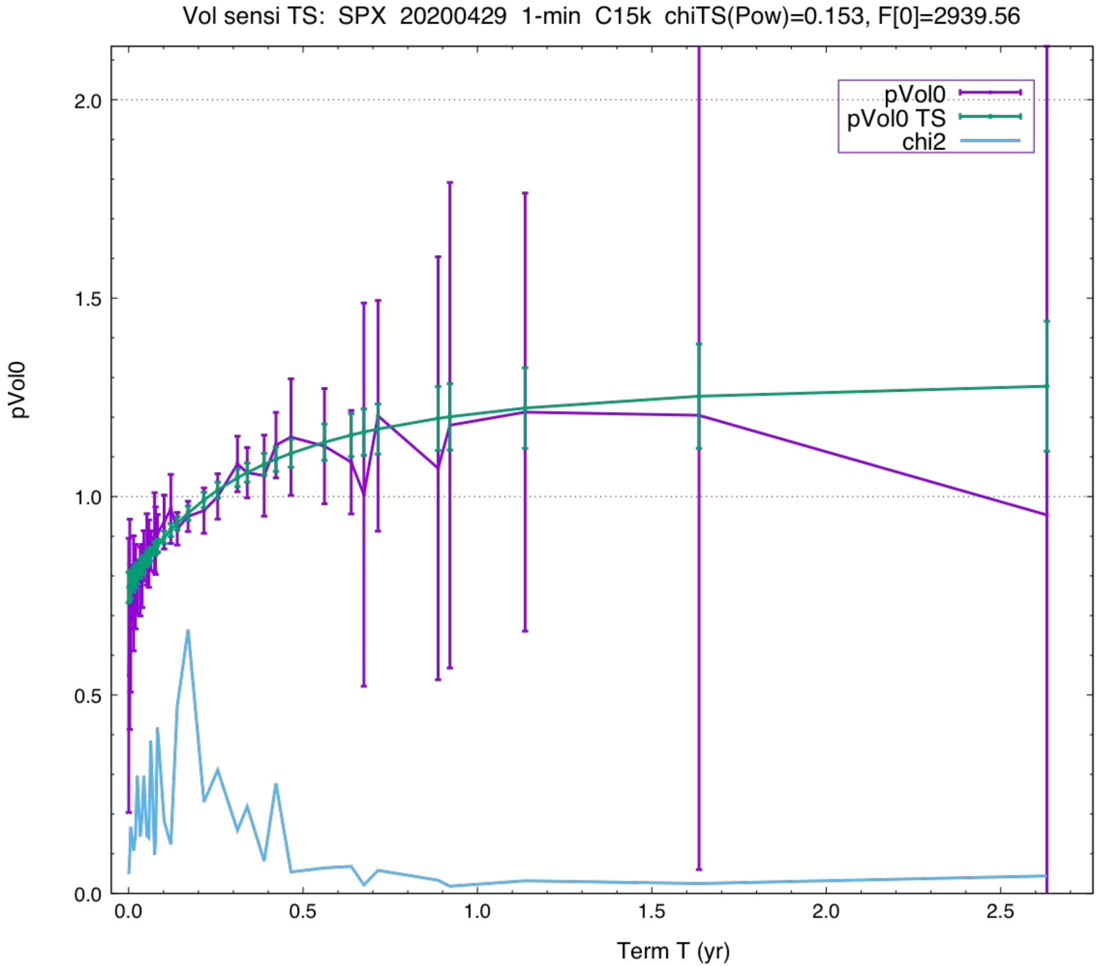
Parametric fit for robustness on small data sets (can be done intra-day)



SPX 20200224

Vol sensitivity (SSR) term-structure

Parametric fit for robustness on small data sets

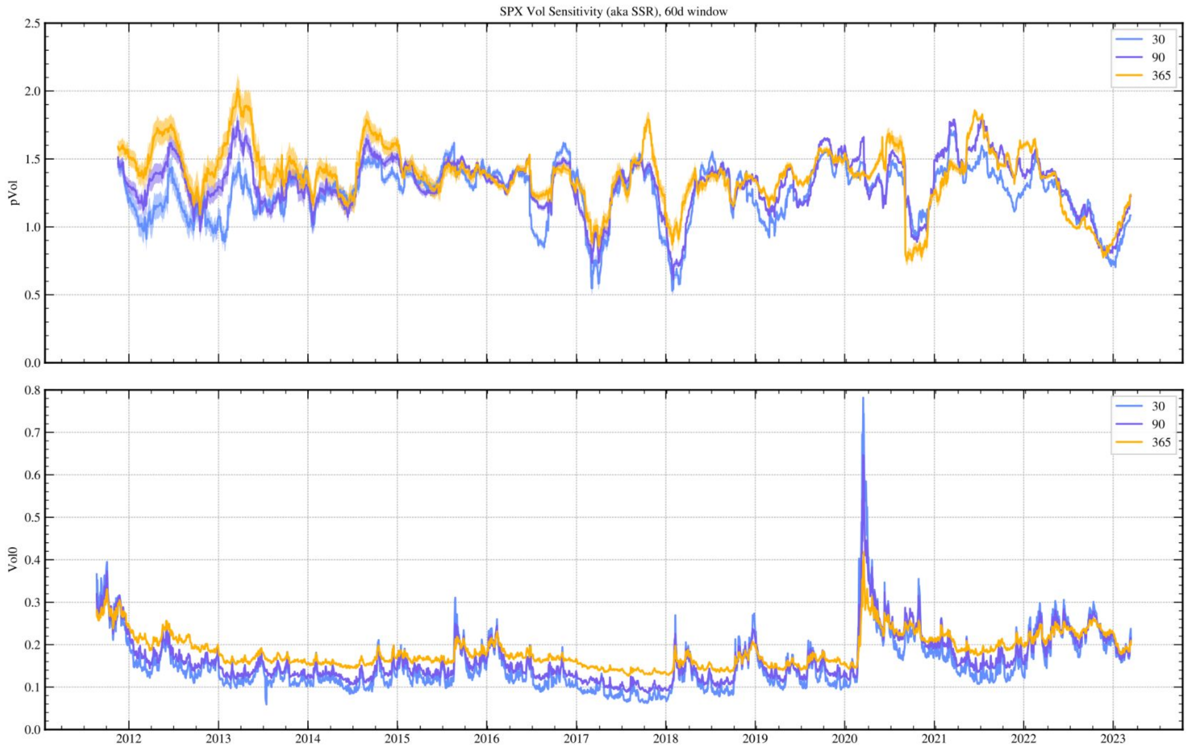


SPX 20200429

Vol sensitivity (SSR) term-structure

On up-days can be upward-sloping, and SSR < 1 at least for some terms

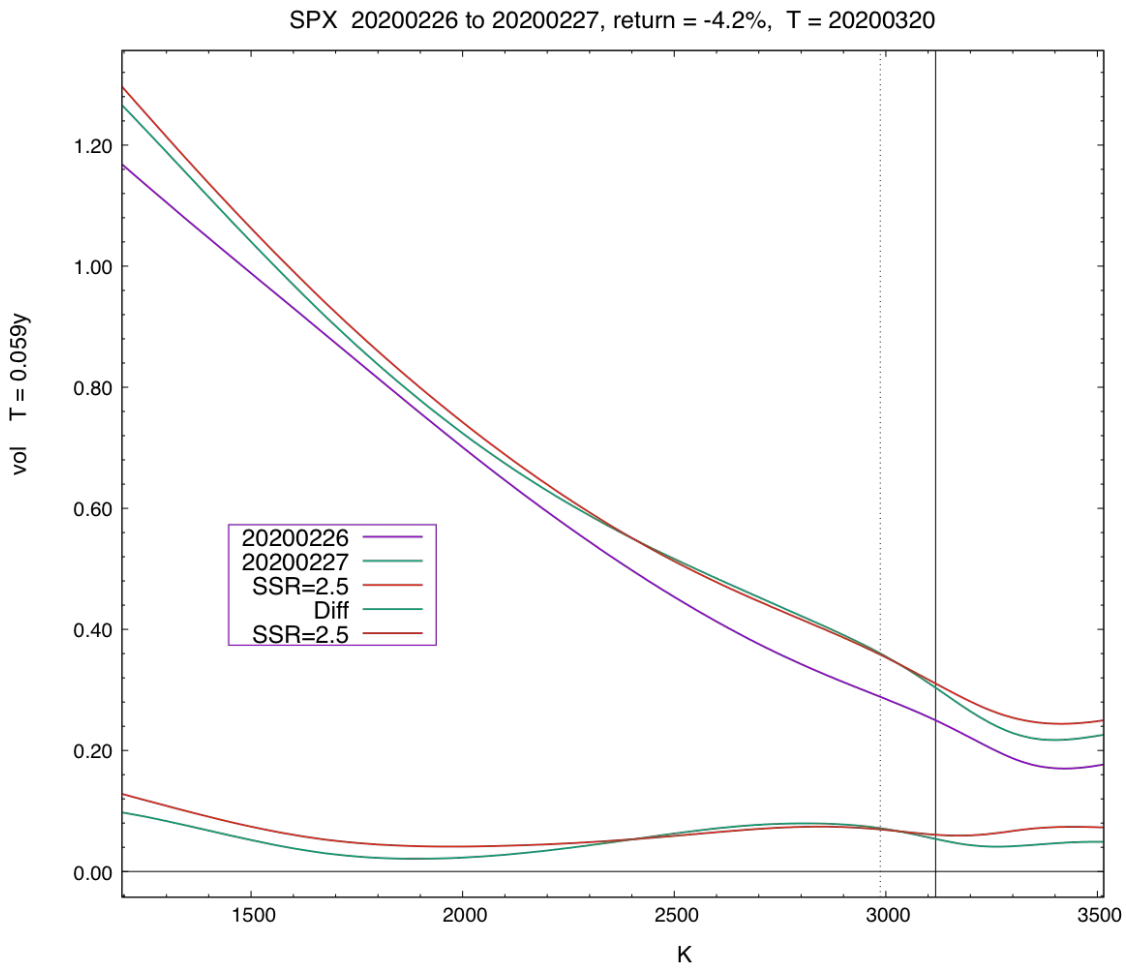
SPX SSR Time-Series 2012 – 2023



Expiries: 30, 90, 365 days

60d trailing window average of close-to-close SSR

Notice low SSR in 2022

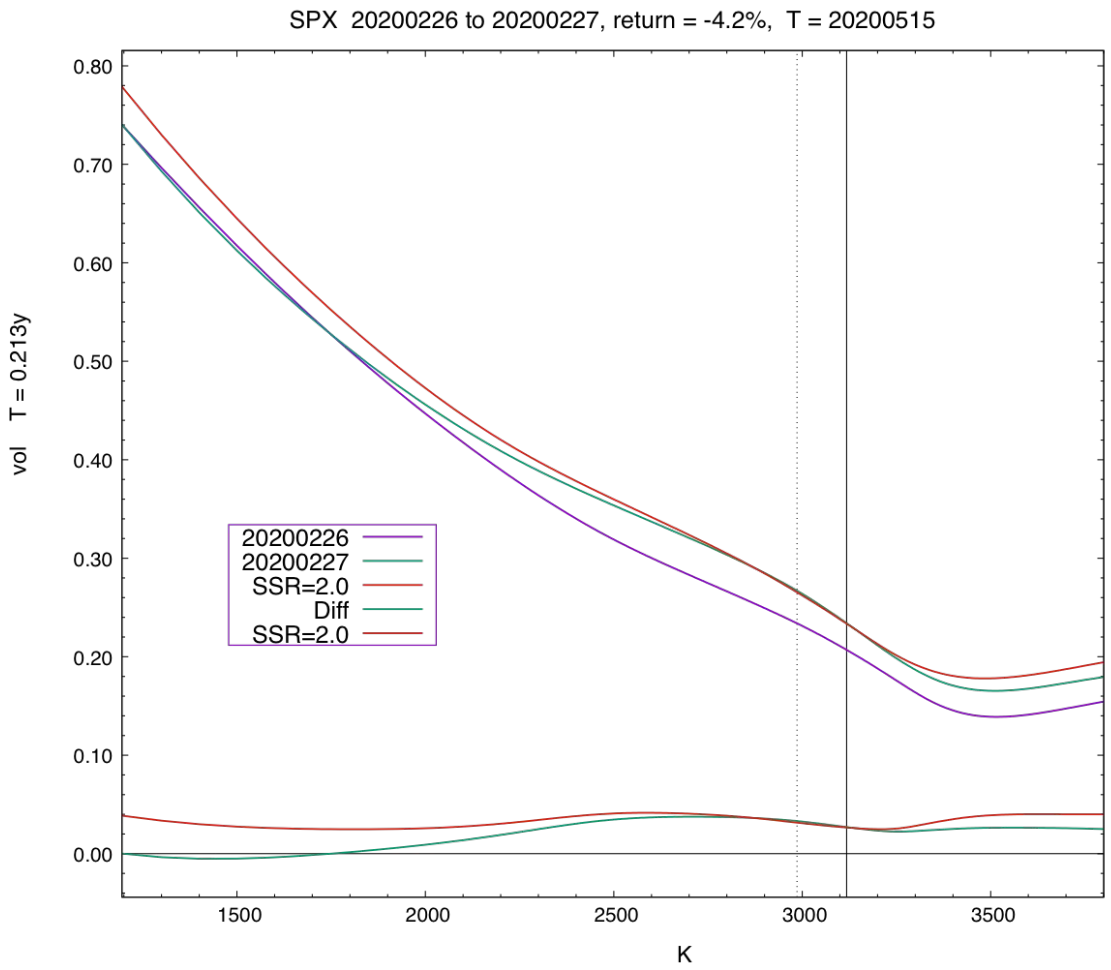


Close-to-close spot vol dynamics

SPX 2020-02-26 to 2020-02-27

T = 3w, SSR = 2.5

Evidence for c2-spot-sensitivity > 0

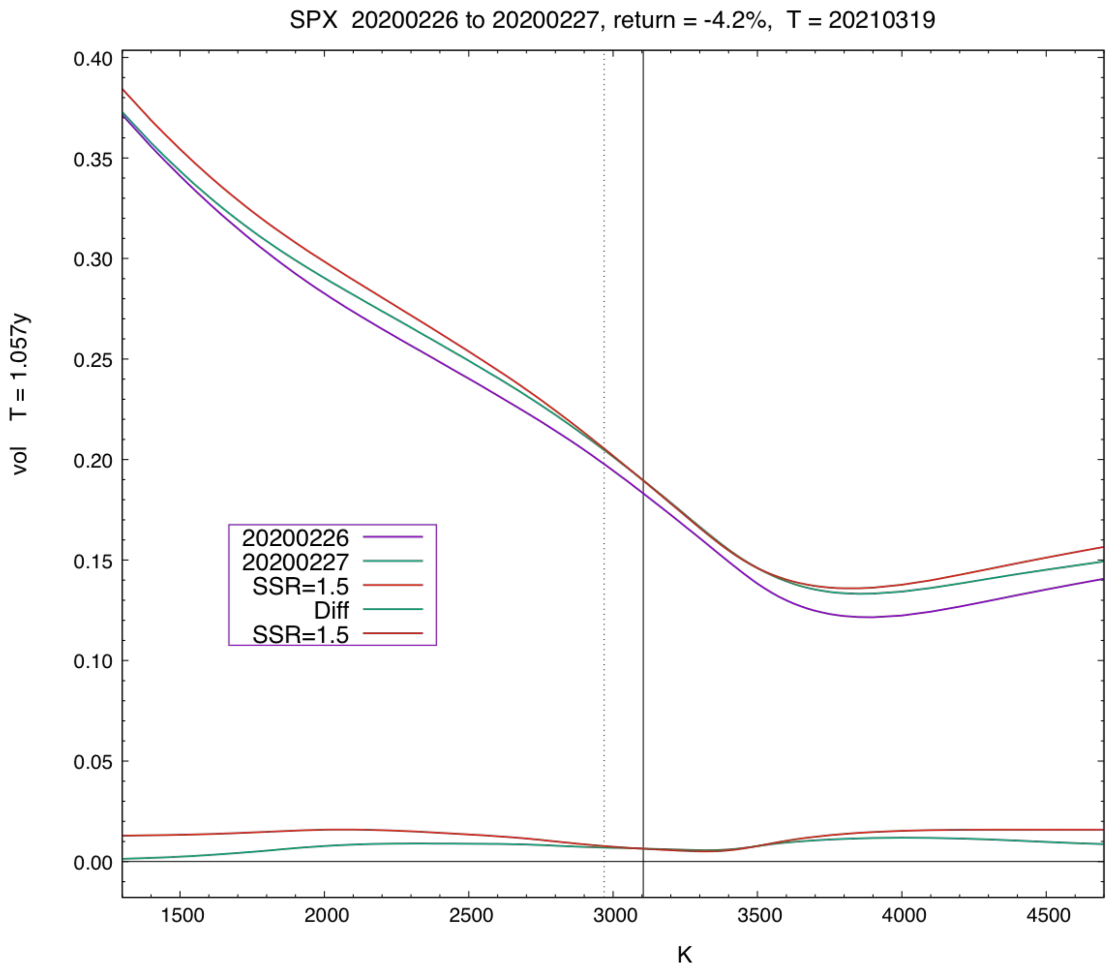


Close-to-close spot vol dynamics

SPX 2020-02-26 to 2020-02-27

T = 2.5m, SSR = 2.0

Evidence for c2-spot-sensitivity > 0

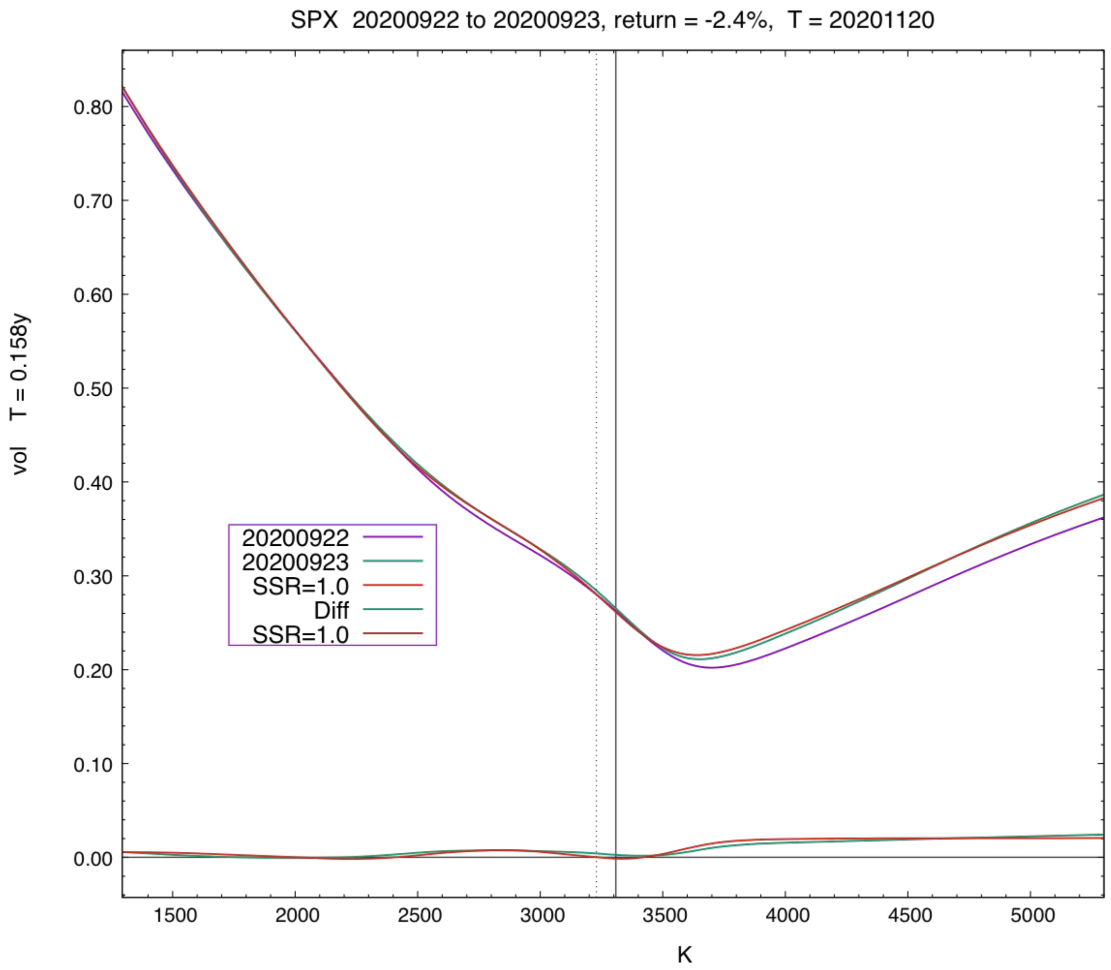


Close-to-close spot vol dynamics

SPX 2020-02-26 to 2020-02-27

T = 1y, SSR = 1.5

Evidence for c2-spot-sensitivity > 0



Close-to-close spot vol dynamics

SPX 2020-09-22 to 2020-09-23

Even when SSR = 1: no sticky strike in the wing(s):

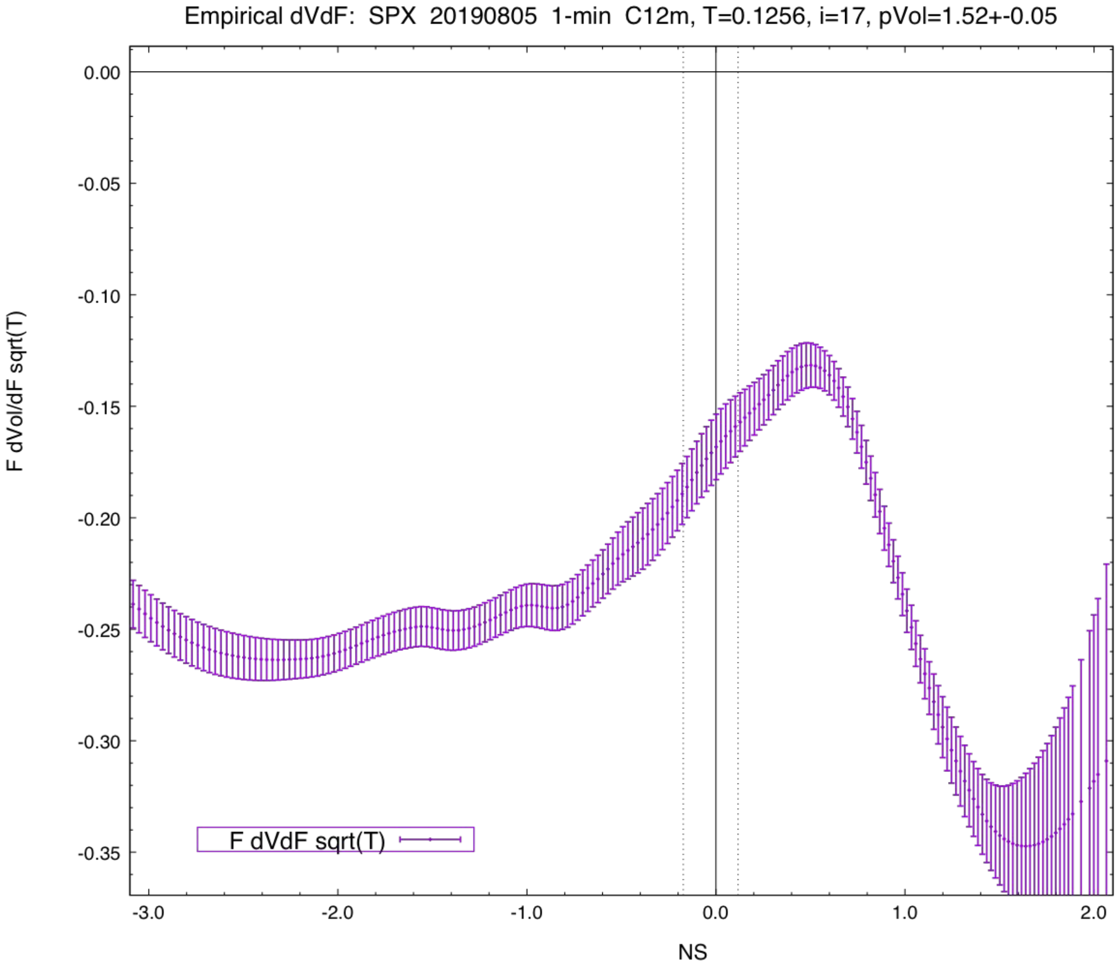
Instead: Shapes are sticky-by-NS!

This down-day comes after a sequence of (minor) down days, and SSR has mean-reverted/reversed to 1…

Spot-Vol Dynamics, Vol Shapes and Delta

VOLA DYNAMICS
Analytics for Options Trading

- What is the **correct delta of a vanilla option**?
  - Delta = DeltaBS + vega * dVol/dF * dF/dS
- dVdF (:= dVol/dF) and the delta adjustment are very large these days!
- dVdF can be calculated from the spot-vol dynamics.
  - Spot-Vol Dynamics is equivalent to knowing the optimal delta (hedges spot-correlated vol move).
- If shapes are stable just one dimensionless number (SSR) is needed.
- Fixed-strike dynamics, i.e. dVdF, and vol parameter dynamics (aka “vol path” for first parameter) behave qualitatively very differently (as we saw already)!
  - Only simple (robust) linear regressions are needed for parameter dynamics.
- For details, see our LinkedIn post.... Or briefly below...

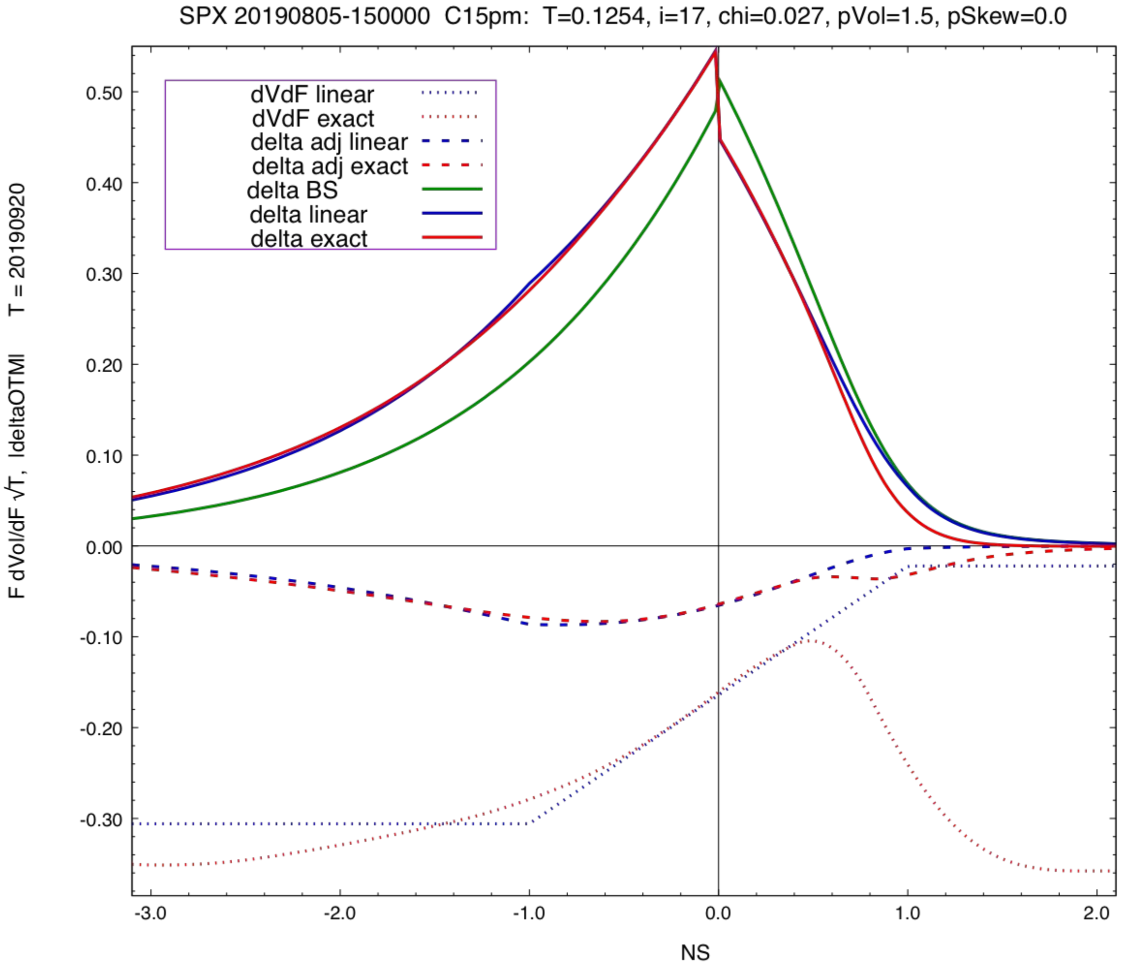


SPX 20190805 T=0.13y M2

Empirical dVdF:

Regression of dvol vs dF for each strike, using 1-min data from 10:00 - 16:00

Note: Fixed strike normalized dVdF is plotted as a function of NS (using average F, T, vol0 over day).



VOLA DYNAMICS
Analytics for Options Trading

SPX 20190805 T=0.13y M2

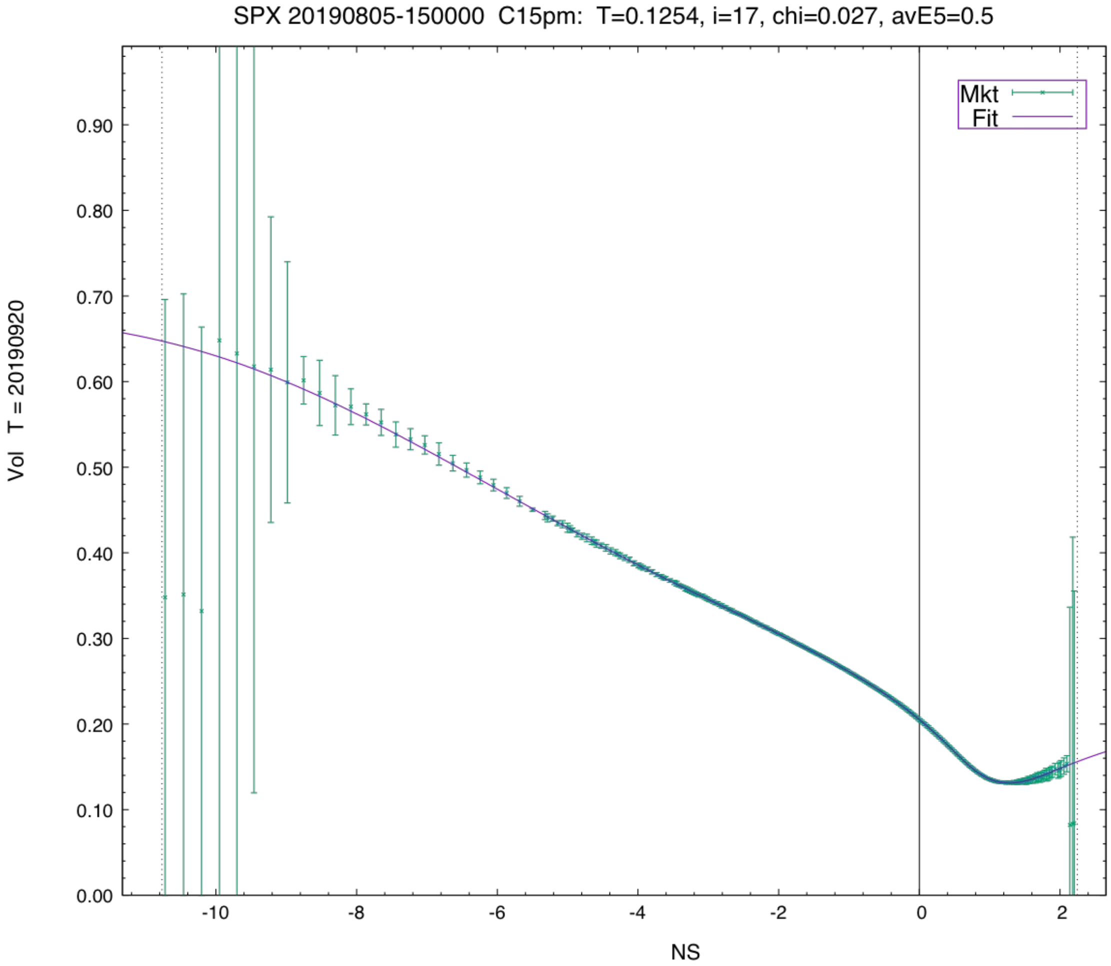
- Normalized dVol/dF
- Delta adjustments
- Final deltas

“Theoretical” dVdF agrees extremely well with empirical dVdF !

These dVdF (etc) curves are extremely stable across time, curve-type, algo details, etc.

Only input: vol fit & SSR (aka pVol) per term.

Some firms use constant or linear approx for dVdF(K):
Linear approx is fine in put wing, bad in call wing



SPX 20190805 T=0.13y M2

Super stable fit….

With steep “knee” at NS = +1.0

ATM parabola does not describe knee at all -- ATM curvature is negative!!

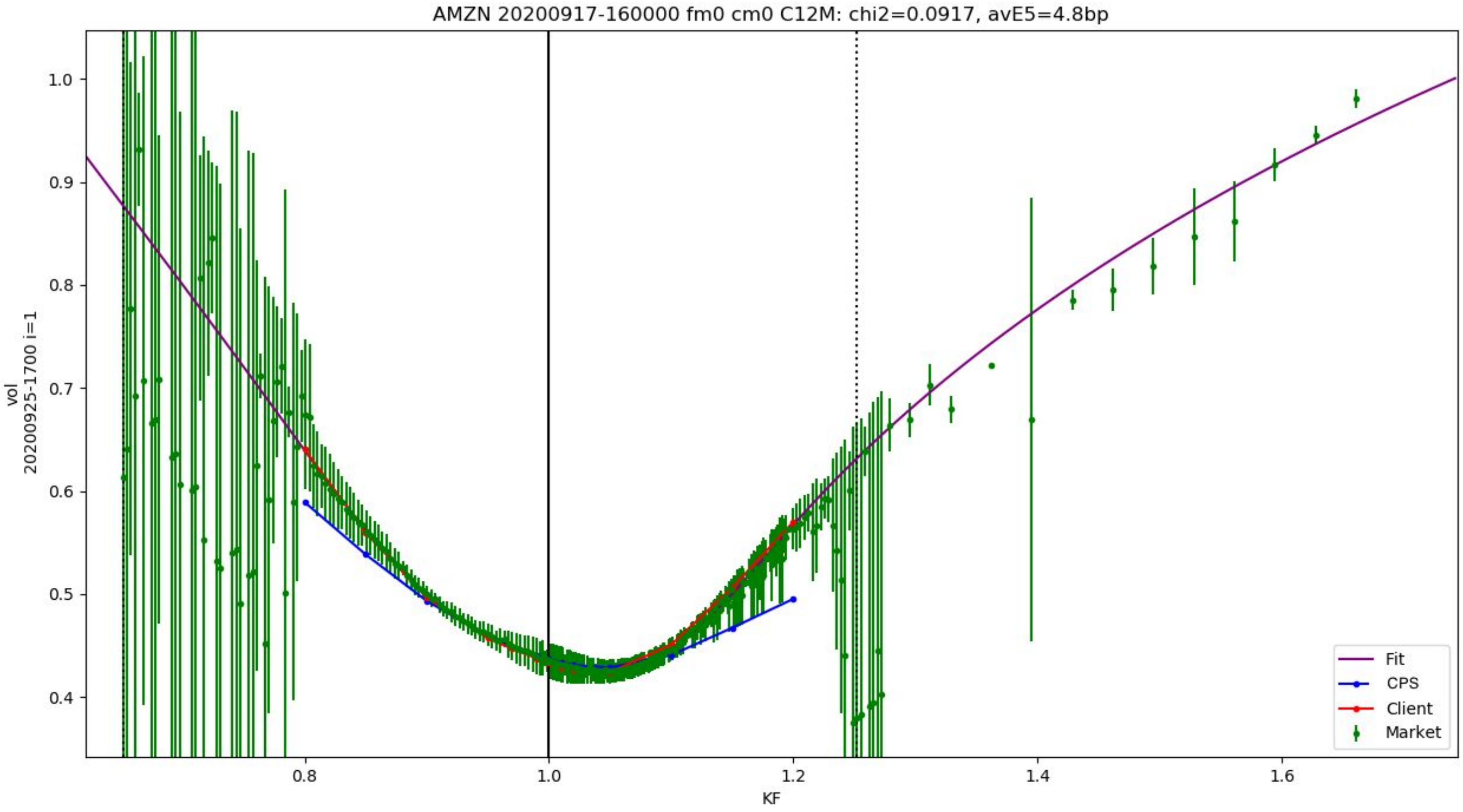
Explains break-down of linear approximation

Questions arising for a bank desk using sub-par curves

- Model Control/IPV & Regulators would like the same surface/theos to be used across Flow, Exotics and OMM desks for a given name (one would hope...)
- How much time is spent massaging curves/surfaces?
  - A lot, it seems. Even then: A top tier bank had no SPX vol surface for 2 days in March 2020...
  - Often not even to match the market (impossible...), but to dampen risk swings...
- If the curves/surfaces are not flexible enough to match the market:
  - Actual "best" fit depends on weights put on different strike ranges. Not stable, will sometimes jump.
  - How to (bias-) correct? Different recipes for each product...
  - Even for var swaps: Is infinite-strip fair vol accurate? No. Is basis stable? Unlikely...
- Structured Products: Simple curves do not even match longer term market...
  - How to hedge with vanillas? How to test that using simple curves for longer-dated SP does not lead to significant model error in valuation and risk? What happens once products are close to expiry?
- How important is proper spot-vol dynamics for exotics/SP deltas, vegas, etc?
- Can one trust a consensus pricing service for options valuation?

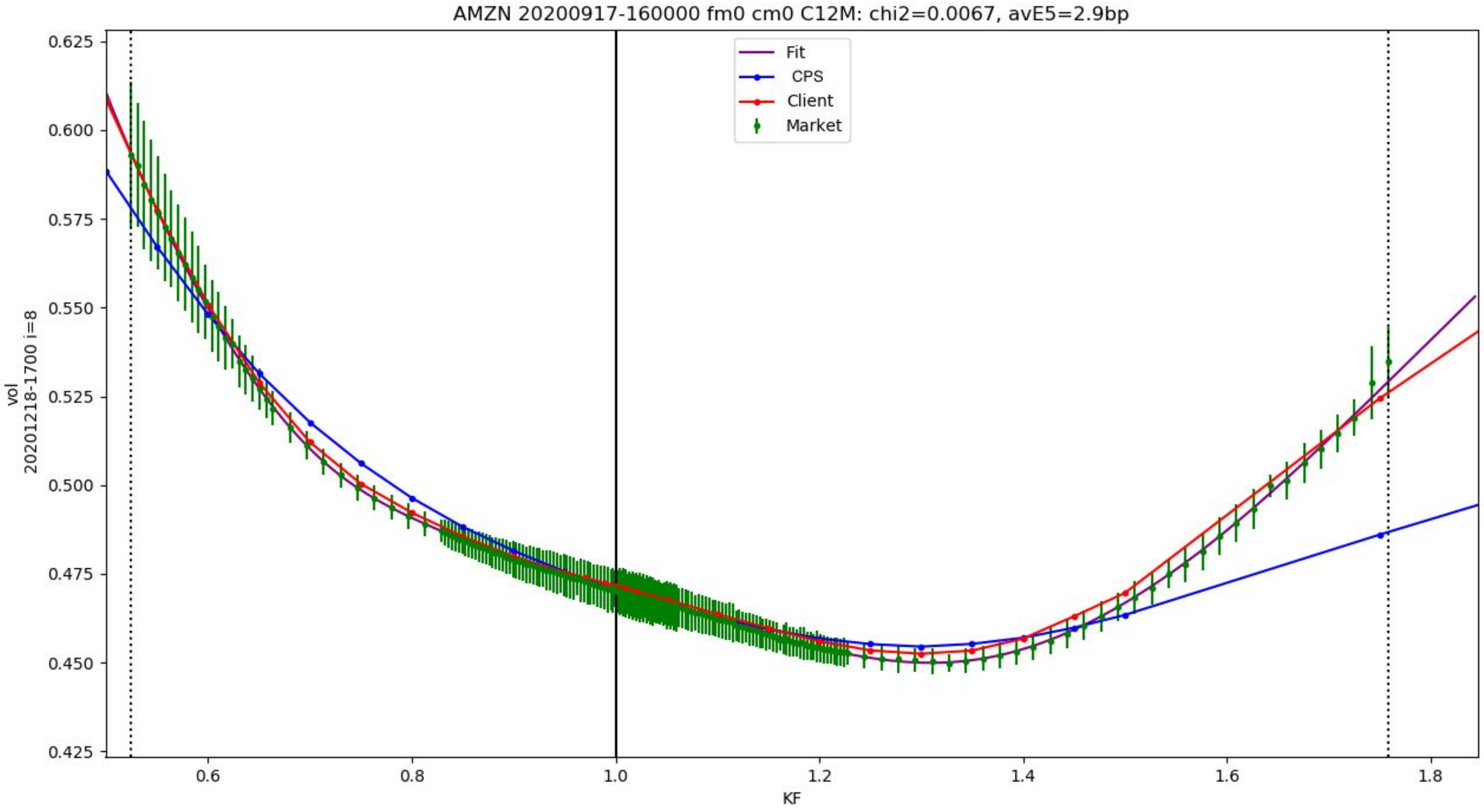
Consensus Pricing Service versus the listed AMZN market

AMZN 2020-09-17, T = 1w



Consensus Pricing Service versus the listed AMZN market

AMZN 2020-09-17, T = 3m



Subtleties of Pricing American “vanillas”

VOLA DYNAMICS
Analytics for Options Trading

- In the olden days:
  - Could price every vanilla, European of American, with one flat r, q, and vol.
  - The same vol would work (well enough…) for call and put at same T,K.
- Already pretty hard, especially in real time. One needs:
  - A proper cash dividend model (no consensus even for vanilla…).
  - Handle settlement effects (incl. exchange and bank holidays).
  - A good choice of “vol time” (aka “business time”), including “events”.
    - NOTE: Pricing with vol time is equivalent to pricing with a (particular) vol term-structure.
  - Then: imply “SPIBOR” (~daily), borrows (real time), and vol surfaces (real time).
  - “American PCP” condition to imply borrow: Demand volP(K) = volC(K) around ATM

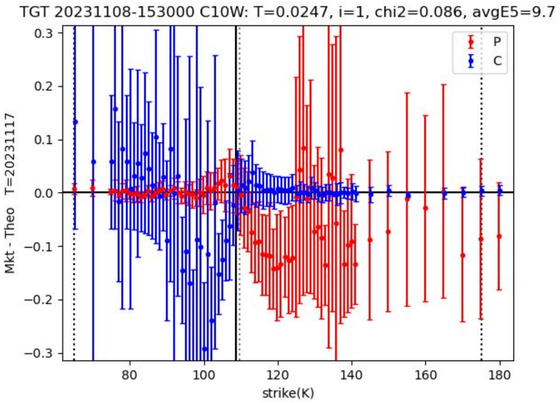
Subtleties of Pricing American "vanillas" 2

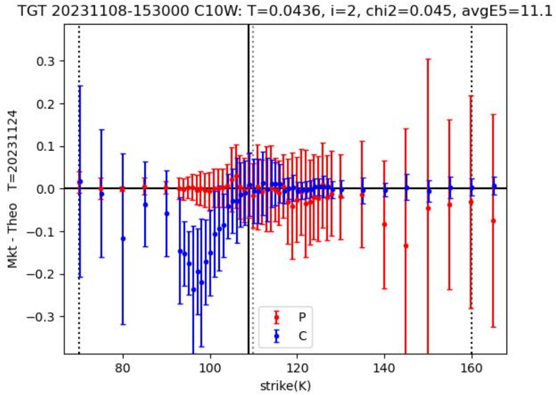
- Now: How fancy does the modeling have to be? ("De-Americanization")

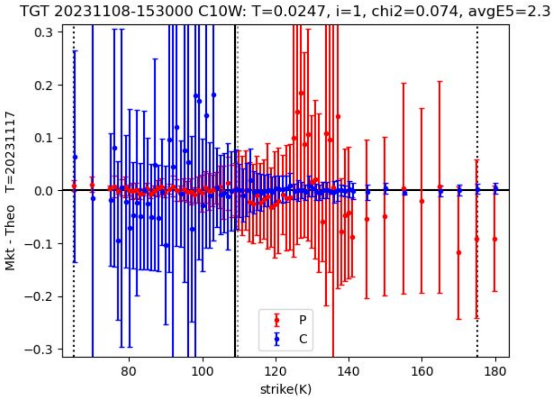
BS:	(1) Flat r,q, vol	(2) r(t), q(t), vol	(3) r(t), q(t), vol(t) for each K(?)
Beyond-BS:	(4) r(t), q(t), LV,	(5) r(t), q(t), SLV,	(6) Other (approx/hacks...)

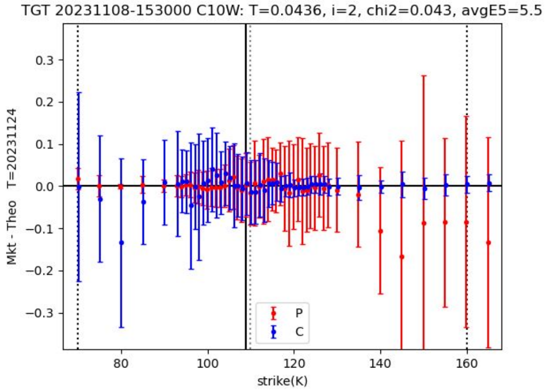
- Empirically in US: One definitely needs rate TS, vol-time including events, settlement, proper dividend modeling.
- In Europe: Evidence that local vols (or roughly equiv approx's) are being used.
- Let's look at some examples:
    - Rate TS and event effects: MSFT, TSLA, TGT
    - Settlement effects (+more): SPX

Event Time Effect on Pricing American Vanillas









TGT 2023-11-08

Target has a dividend and earnings call just before expiry T=2023-11-17 (i=1).

Top row: Without an "event time" an implied borrow allows (OTM and ITM) market prices to be matched at a few strikes, but not all.

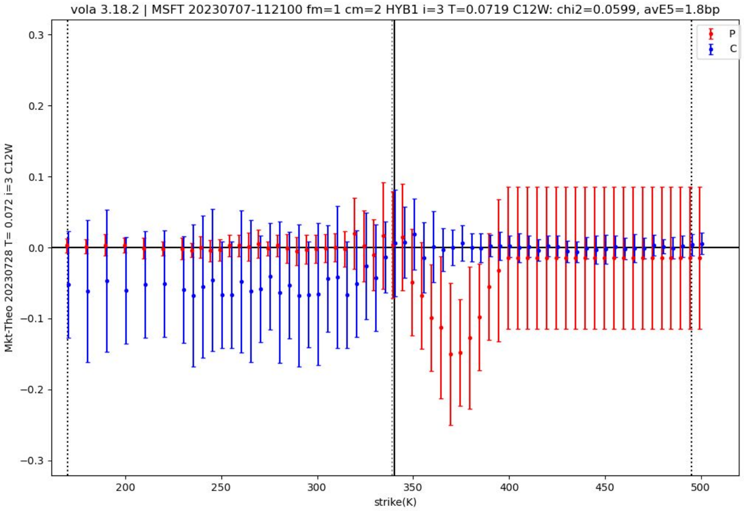
Bottom row: With an event time of 0.09y all prices can be matched, in all expiries!

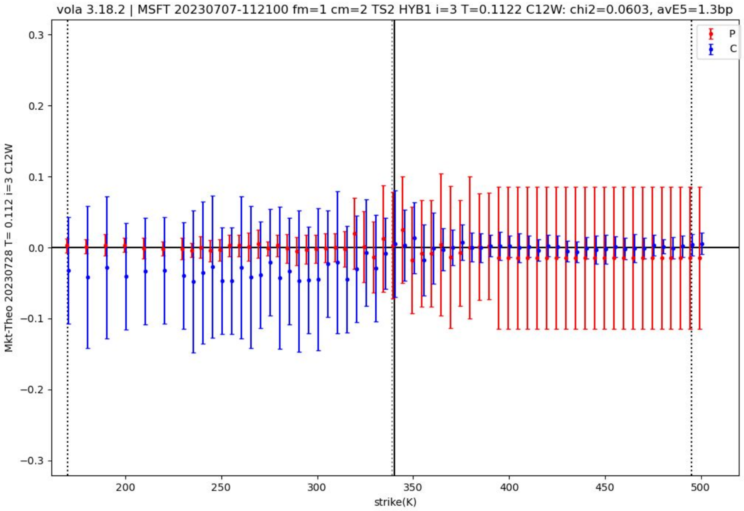
Rate TS and Event Time for American Vanillas

MSFT 2023-07-07

The ultimate test of a valuation approach is always the price-difference plot: Mkt - Theo

Flat term rates r(T), q(T)	Local r(t),q(t) and $\Delta T_E = 0.04y$

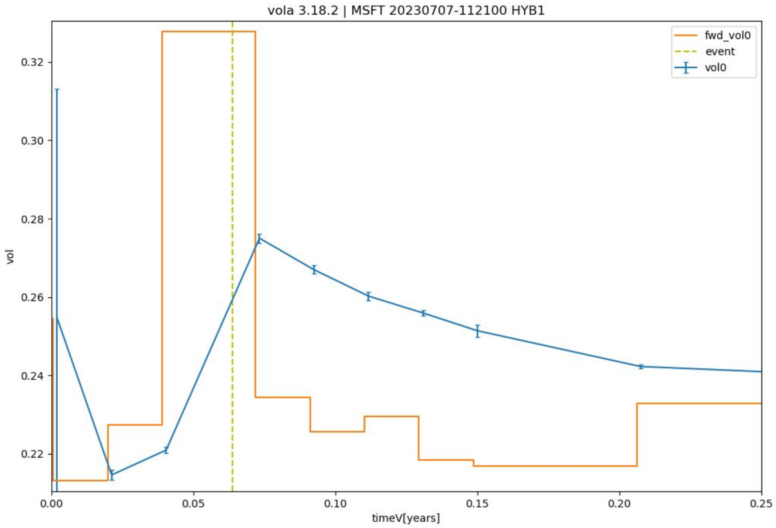




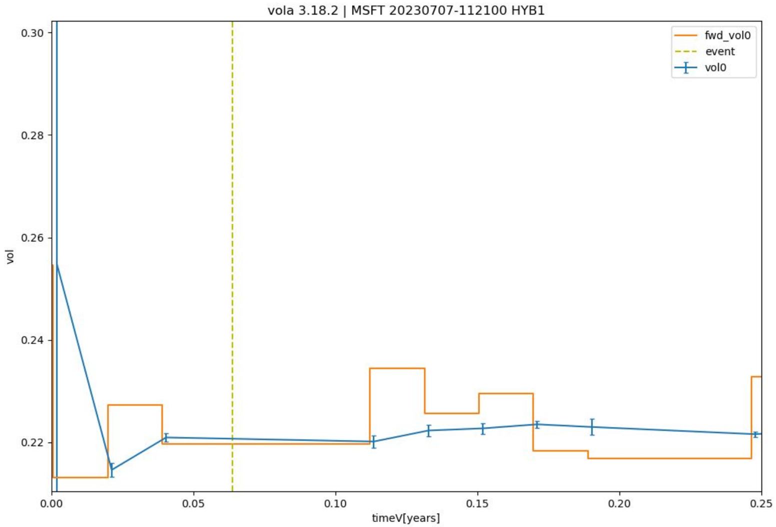
Rate TS and Event Time for American Vanillas

MSFT 2023-07-07

“Dirty” ATF vols



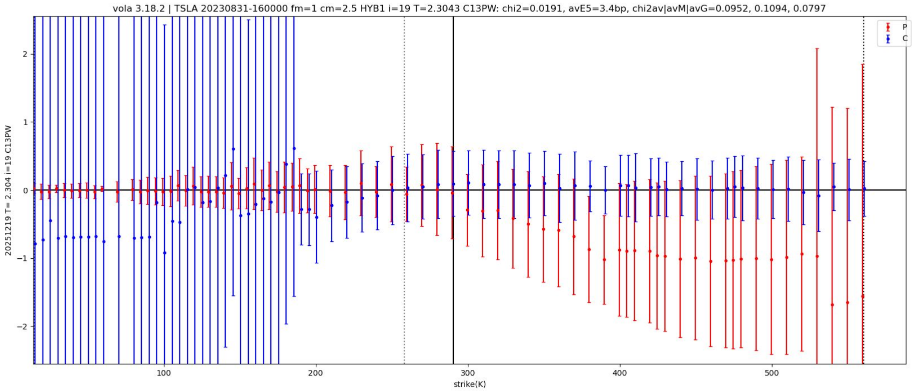
“Clean” ATF vols, $\Delta T_E = 0.04y$



Rate Term-Structure Effect on Pricing American Vanillas

VOLA DYNAMICS
Analytics for Options Trading

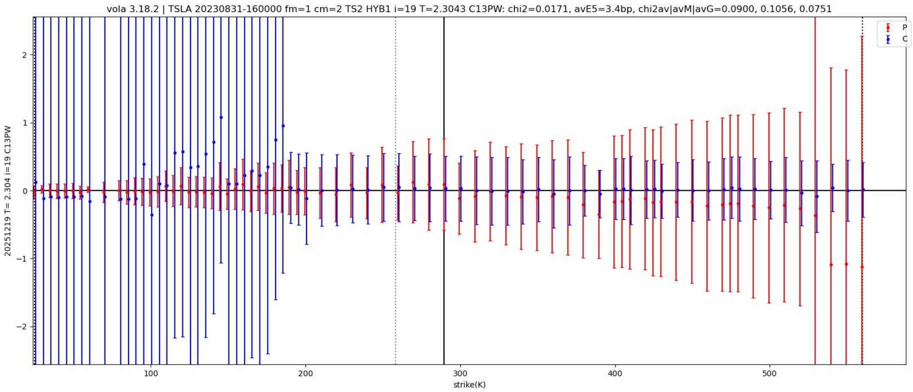
TSLA 2023-08-31



Price-Difference plot: Mkt - Theo

← Pricing with flat term r,q

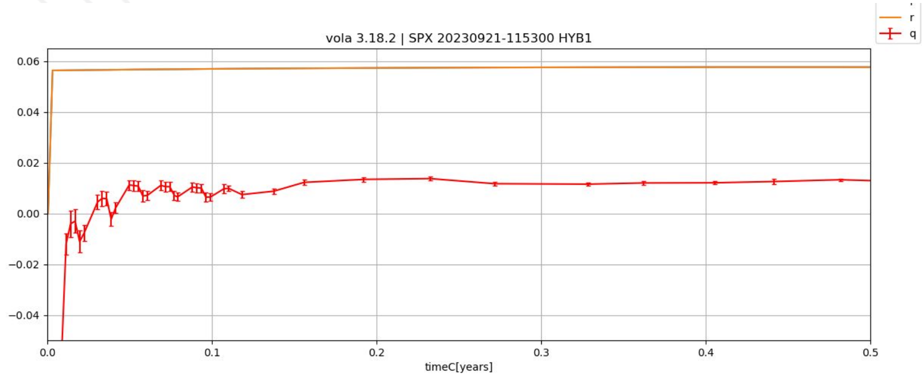
T = 2.4y



← Pricing with local r(t),q(t)

Settlement Effects for SPX options

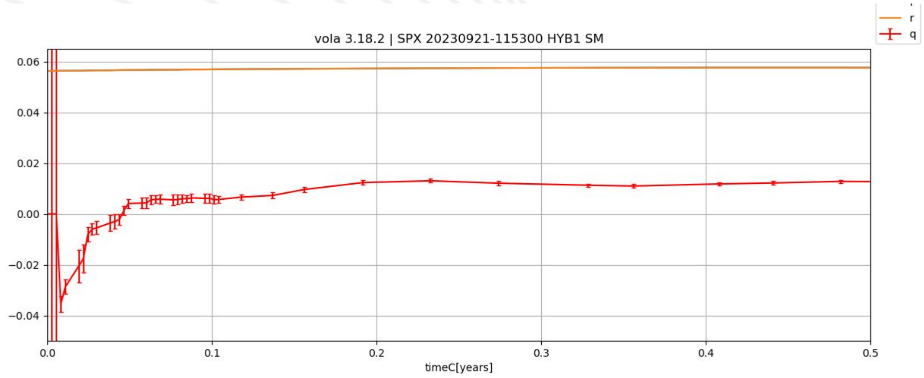
Let's treat SPX like an equity with a "spot", borrow cost, and (perhaps) cash dividends.



Implied borrow cost term structure

← Ignoring settlement, wrong spot

Wrong spot shows up as 1/T term in the borrow TS (made up wrong spot for illustration here…).

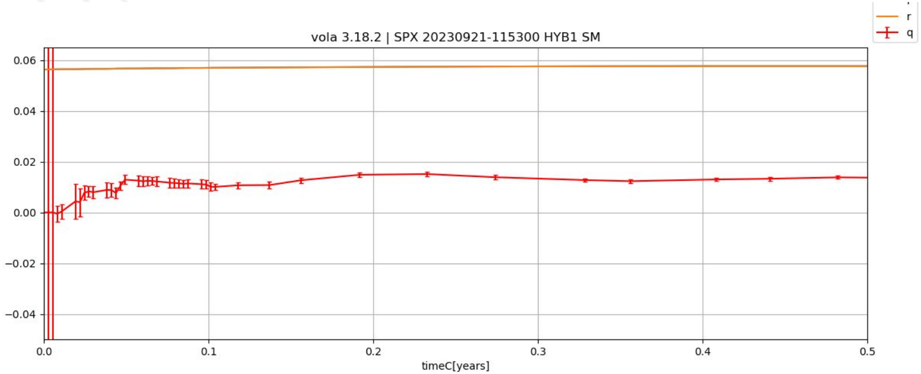


← With settlement, wrong spot

Now short-term borrow TS is smooth.

Settlement Effects for SPX options

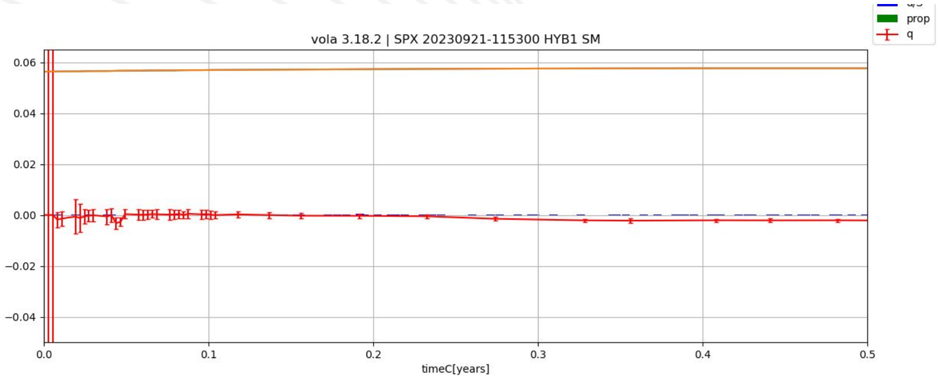
Let's treat SPX like an equity with a "spot", borrow cost, and (perhaps) cash dividends.



Implied borrow cost term structure

← With settlement, implied spot

No divs, so borrow includes div yield



← With settlement, implied spot

With divs, so borrow is "pure" and very flat close to 0

What we didn’t talk about!

- Details of implied borrows, forwards.
- Fine control of fits, e.g. temporal filtering, priors.
- Easy, realistic scenarios.
- PnL explain in terms of greek or factors (spot, vol, skew,...)
- Vol derivatives pricing, consistent greeks with vanillas.
- VIX futures relationship to SPX and VIX vol surfaces.
- Non-Equity underliers.

Questions?

Stop by the Vola Dynamics booth for more fun!

- Sophisticated banks, hedge funds and prop shops rely on the Vola Dynamics quant library.
- See VolaDynamics.com, email info@VolaDynamics.com